A- SOLOW-SWAN MODEL

- Model assumes constant saving rate and aims to analyze:
  - The steady state of the economy (given the saving rate)
    - where steady state defines situation where variables (capital, output, and so on) grow at constant rate
  - The optimal saving rate that maximizes consumption
  - Transitional dynamics while economy converges to its steady state
  - How different levels of saving can explain income dispersion across countries

Model Specifications

- Closed Economy
- Neo-classical production function (Y(t)=F(K(t),L(t))
  - where physical capital (K) and labor (L) are inputs of production

  - for now we do not use technological process in the production function

- Capital evolves according to the process

$$\dot{K} = sF(K, L) - \delta K$$

  - where s is the saving rate (fraction of output that is saved in the economy) and δ is depreciation rate of capital
  - This equation states that $\dot{K}$ is the net increase in stock of physical capital and explained by gross investment, $sF(K, L) = I$, minus depreciated capital, δK
    - *Assumption:* s is given exogenously
  - Consumption is equal to (time indexes are suppressed)

$$C = F(K, L) - I = (1 - s)F(K, L)$$

- Population grows at a constant rate, $\frac{\dot{L}}{L} = n \quad \text{i.e. } L_t = L_0 e^{nt}$

Neoclassical Production Function A production function is neoclassical if it exhibits

- Positive and diminishing marginal product, $\forall K > 0 \;\; \& \;\; \forall L > 0$,

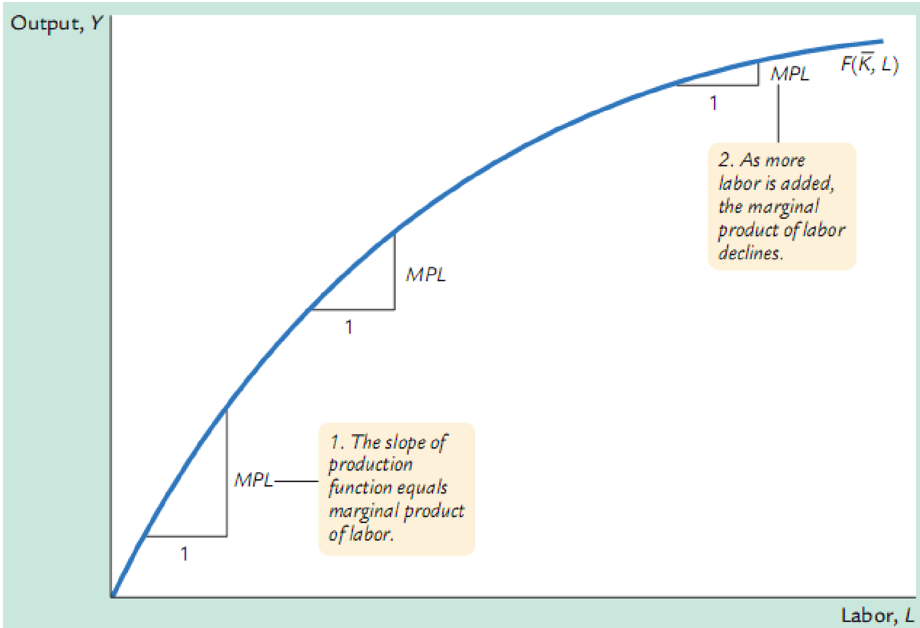
$$\frac{\partial F}{\partial K} > 0 \qquad \frac{\partial^2 F}{\partial K^2} < 0$$
$$\frac{\partial F}{\partial L} > 0 \qquad \frac{\partial^2 F}{\partial L^2} < 0$$

- Inada Conditions

$$\lim_{K \to 0} F_K = \infty \qquad \lim_{L \to 0} F_L = \infty$$
$$\lim_{K \to \infty} F_K = 0 \qquad \lim_{L \to \infty} F_L = 0$$

- CRTS (constant return to scale property): $F(\lambda K, \lambda L) = \lambda F(K, L)$

Graphical Explanation of Neoclassical Production Function



The Production Function This curve shows how output depends on labor input, holding the amount of capital constant. The marginal product of labor *MPL* is the change in output when the labor input is increased by 1 unit. As the amount of labor increases, the production function becomes flatter, indicating diminishing marginal product.

Ex: Cobb-Douglas Production Function $Y_t = K_t^{\alpha} L_t^{(1-\alpha)}$ where $0 < \alpha < 1$

Marginal product is positive but diminishing

$$\frac{\partial F}{\partial K} = \alpha K_t^{\alpha-1} L_t^{1-\alpha} > 0 \qquad \frac{\partial^2 F}{\partial K^2} = \alpha(\alpha-1) K_t^{\alpha-2} L_t^{1-\alpha} < 0$$

$$\frac{\partial F}{\partial L} = (1-\alpha) K_t^{\alpha} L_t^{-\alpha} > 0 \qquad \frac{\partial^2 F}{\partial L^2} = -(1-\alpha)\alpha K_t^{\alpha} L_t^{-\alpha-1} < 0$$

Inada Conditions are satisfied

$$\lim_{K\to 0} F_K = \lim_{K\to 0} \alpha K_t^{\alpha-1} L_t^{1-\alpha} = \infty \qquad \lim_{L\to 0} F_L = (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = \infty$$

$$\lim_{K\to\infty} F_K = \lim_{K\to 0} \alpha K_t^{\alpha-1} L_t^{1-\alpha} = 0 \qquad \lim_{L\to\infty} F_L = (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = 0$$

CRTS

$$F(\lambda K, \lambda L) = (\lambda K_t)^{\alpha} (\lambda L_t)^{(1-\alpha)} = \lambda^{\alpha} \lambda^{(1-\alpha)} K_t^{\alpha-1} L_t^{1-\alpha} = \lambda F(K, L)$$

As a result Cobb Douglas satisfies properties of neoclassical production function

Writing Variables in Terms of Per Capita

- CRTS production function can also be written as

$$Y = F(K, L) \quad \Rightarrow \quad Y/L = F(K/L, L/L) = F(K/L, 1)$$

  – by definition Y/L is output per labor (or per capita) and K/L is capital per labor. If we denote them by y and k : $y = f(k)$

  – k is useful to compare large and small countries as it is in per capita terms

- Marginal products of capital and labor can be found in terms of k.

  – Given that $Y = LF(K/L, 1)$,

$$\frac{\partial Y}{\partial K} = LF_{K/L}(\frac{K}{L}, 1)\frac{\partial (K/L)}{\partial K} = LF_{K/L}(\frac{K}{L}, 1)\frac{1}{L} = f'(k) > 0$$

$$\frac{\partial Y}{\partial L} = F(\frac{K}{L}, 1) + LF_{K/L}(\frac{K}{L}, 1)(-\frac{K}{L^2}) = f(k) - f'(k)k > 0$$

If $F()$ is a neoclassical production function, $f()$ satisfies properties of $F()$ as well

Ex: Cobb-Douglas Production Function $Y_t = K_t^\alpha L_t^{(1-\alpha)}$ where $0 < \alpha < 1$

$$\frac{Y_t}{L_t} = K_t^\alpha L_t^{-\alpha} = (k_t)^\alpha \qquad \Rightarrow \qquad y_t = k_t^\alpha$$

- Marginal product is positive but diminishing

$$f'(k) = \alpha k_t^{\alpha-1} > 0$$
$$f''(k) = \alpha(\alpha-1)k_t^{\alpha-2} < 0$$

- Inada Conditions are satisfied

$$\lim_{k\to 0} \alpha k_t^{\alpha-1} = \infty \qquad \Rightarrow \qquad \lim_{k\to 0} f'(k) = \infty$$
$$\lim_{k\to \infty} \alpha k_t^{\alpha-1} = 0 \qquad \Rightarrow \qquad \lim_{k\to \infty} f'(k) = 0$$

Dynamic Equation for Capital Stock

The following dynamic equation

$$\dot{K} = sF(K, L) - \delta K$$

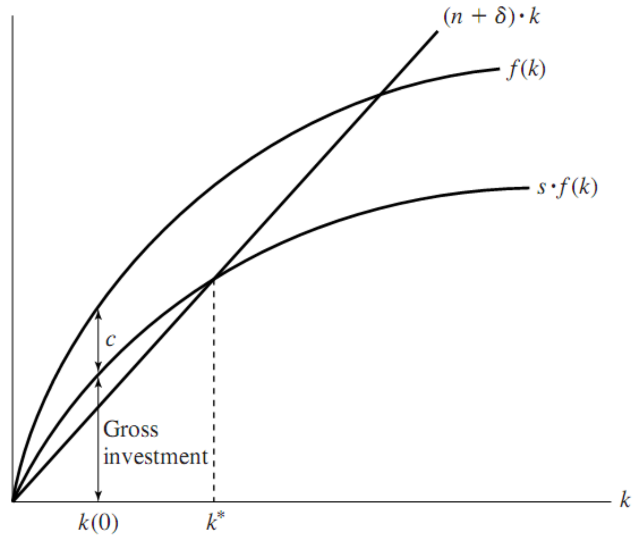
can be written in terms of k and $\dot{k}$, where $\dot{k}$ can be found by

$$\dot{k} = (\frac{\dot{K}}{L}) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{sF(K, L)}{L} - \delta k - nk$$

$$\Rightarrow \dot{k} = sf(k) - (\delta + n)k$$

$$\dot{k} = sf(k) - (\delta + n)k$$

This is the fundamental equation of Solow-Swan, and $(\delta + n)$ is the effective depreciation rate



Steady State (k^*)

- Steady state is the situation where various quantities grow at a constant rate (like constant $\frac{\dot{k}}{k}$)

- For the previous model it can be shown that at the steady state,

$$\frac{\dot{k}}{k} = 0, \quad \text{and} \quad \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = n$$

- **Proof:**

  $\frac{\dot{k}}{k}$: $\quad \dot{k} = sf(k) - (\delta + n)k \quad \Rightarrow \quad \frac{\dot{k}}{k} = \frac{sf(k)}{k} - (\delta + n)$

  We know that in the steady state $\dot{k}/k$ is constant. $(\delta+n)$ and s are also constants by definition. For the last inequality to hold, $f(k)/k$ must be constant as well. This means that the derivative of this term should be 0

$$\frac{d(f(k)/k)}{dt} = \frac{f'(k)\dot{k}k - f(k)\dot{k}}{k^2} = \frac{\dot{k}}{k} * \frac{f'(k)k - f(k)}{k} = \frac{\dot{k}}{k} * \frac{-MPL\ (<0)}{k} = 0.$$

$$\Rightarrow \quad \frac{\dot{k}}{k} = 0.$$

– This further implies that in the Solow-Swan model, the steady state level of capital, k^*, algebraically satisfies $sf(k^*) = (\delta + n)k^*$

$$\frac{\dot{K}}{K}: \; K = kL \quad \Rightarrow \quad \dot{K} = \dot{k}L + k\dot{L} \quad \Rightarrow \quad \frac{\dot{K}}{K} = \dot{k}\frac{L}{K} + k\frac{\dot{L}}{K} = \frac{\dot{k}}{k} + k\frac{\dot{L}}{L}\frac{L}{K} = \frac{\dot{k}}{k} + n = n$$

Golden Rule Level of Capital and Dynamic Inefficiency

- The steady state level of consumption $c^* = (1-s)f(k^*)$
- The saving rate that maximizes the steady state consumption per person is called the *Golden Rule Level of Saving Rate*, and denoted by

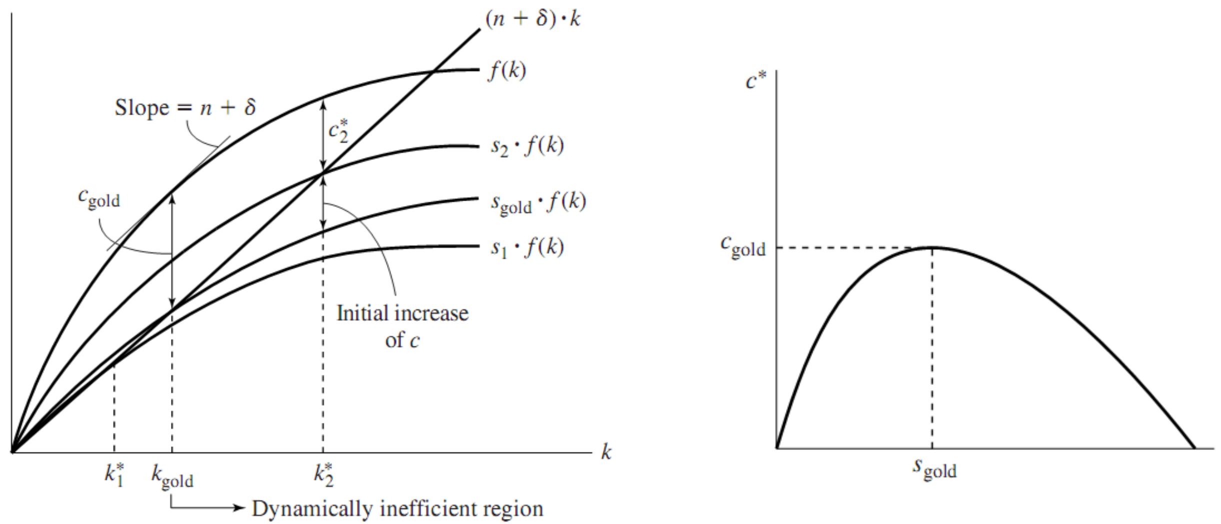
$$s_{gold}: \max_s c^* = \max_s (1-s)f(k^*(s))$$

  - We have shown that at the steady state $sf(k^*) = (\delta+n)k^*$.
  - This implies that

$$\max_s c^* = \max_{k^*}[f(k^*) - (\delta+n)k^*]$$

FOC requires that

$$f'(k_{gold}) = (\delta+n)$$



- $f'(k_{gold}) = (\delta + n)$ implies that it is optimal to accumulate capital till its marginal product equals to the effective depreciation rate. Before (after) this point the marginal product of capital is higher (lower) than the effective depreciation rate.

- If $s > s_{gold}$, it is dynamically inefficient region ($k^* > k_{gold}$ but $c^* < c_{gold}$!)

Transitional Dynamics

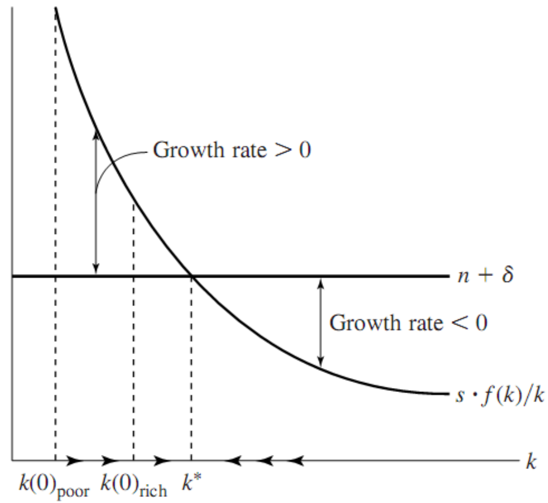
- $\dot{k} = sf(k) - (\delta + n)k$
- $\gamma_k = \dfrac{\dot{k}}{k} = s\dfrac{f(k)}{k} - (\delta + n)$ where γ_k is the growth rate of capital that explains how fast $\dot{k}$ evolves
- We can understand the behavior of γ_k from the derivative of $\dfrac{f(k)}{k}$

$$\frac{\partial(sf(k)/k)}{\partial k} = s\frac{f'(k)k - f(k)}{k^2} = -s\frac{MPL}{k^2} < 0$$

which decreases in k. To find its curvature, we can either look for the second derivative, which is burdersome, or look at its limits

$$\lim_{k \to 0} \frac{sf(k)}{k} = \lim_{k \to 0} sf'(k) = \infty \quad \text{(inada)}$$
$$\lim_{k \to \infty} \frac{sf(k)}{k} = \lim_{k \to \infty} sf'(k) = 0 \quad \text{(inada)}$$

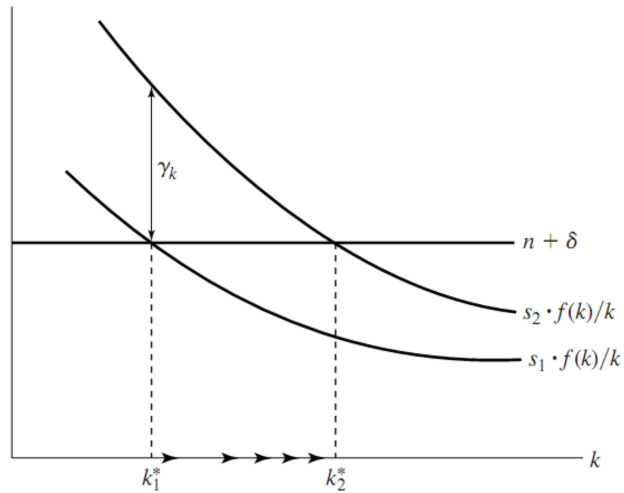
- This can be shown by the following figure



Notice that if $k < k^*$, then $\gamma_k > 0$. Also if $\gamma_k \to 0$ as $k \to k^*$

The source of this result is diminishing return to capital: when k is relatively small, then average product of capital $f(k)/k$ is relatively high, and so does $sf(k)/k$

Policy Experiment: The effect of an increase in the saving rate



- Say at steady state, k_1^*, a government policy induces a saving rate s_2 that is permanently higher than s_1. Then $\gamma_{k_1^*} > 0$. But as $k \uparrow$, $\gamma_k \downarrow$ and approaches 0. Thus, a permanent increase in saving rate generates a temporarily positive per capita growth rates

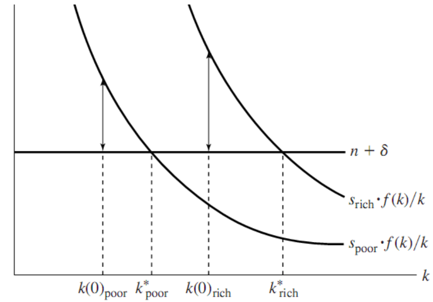
Absolute and Conditional Convergence

Smaller values of k are associated with larger values of γ_k

Question: Does this mean that economies of lower capital per person tend to grow faster in per capita terms? Is there a tendency for convergence across economies?

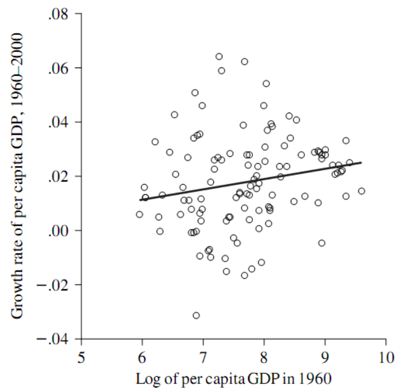
- *Absolute Convergence:* The hypothesis that poor countries grows faster per capita than rich ones without conditioning on any other characteristics of the economy
- *Conditional Convergence:* If one drops the assumption that all economies have the same parameters, and thus same steady state values, then an economy grows faster the further it is own steady state value
- Ex: consider two economies that differ only in two respects: $s_{rich} > s_{poor}$ & $k(0)_{rich} > k(0)_{poor}$
  - Does the Solow model predict poor economy will grow faster than the rich one?

∗ Not necessarily. See the figure:

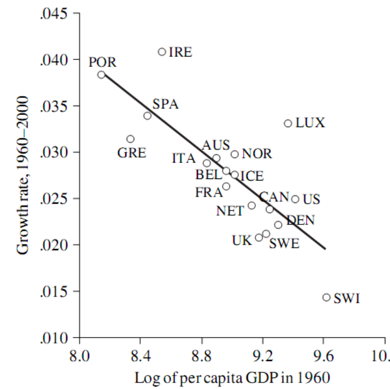


∗ Data:

Absolute Convergence
Sample of 114 countries



Conditional Convergence
Sample of OECD countries



Technological Progress

- In the absence of technological progress, diminishing returns makes it impossible to maintain per capita growth for so long just by accumulating more capital per worker

- There are ways of modelling technological progress

  - *Hicks Neutral:* $Y = T(t)F(K, L)$

  - *Harold Neutral (Labor Augmenting):* $Y = F(K, A(t)L)$

  - *Solow Neutral (Capital Augmenting):* $Y = F(B(t)K, L)$

    * It is shown that only labor-augmenting progress is consistent with existence of a steady state (that is with constant growth rates) in neoclassical growth modelsh unless the production function is a Cobb Douglas, for which the steady state is obtained with any form of technological process.

Solow-Swan Model with Labor Augmenting Technological Growth

- Assume the production function is labor augmenting and $\frac{\dot{A}}{A} = x$. The fundamental equation of the model implies that

$$\dot{K} = sF(K, LA) - \delta K$$

$\dot{k}$ can be found by

$$\dot{k} = (\frac{\dot{K}}{L}) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{sF(K, A(t)L)}{L} - \delta k - nk$$

- We know that at steady state γ_k is constant

$$\gamma_k = \frac{\dot{k}}{k} = sF(1, \frac{A}{k}) - (\delta + n)$$

this equation states that $F(1, A/k)$ is constant, meaning that k and A grows at the same rate, which is $\gamma_{k^*} = x$

– Since k and A grows at the same rate in the steady state, define

$$\hat{k} = \frac{k}{A} = \frac{K}{LA} \quad \text{where } \hat{k} \text{ is capital per effective unit labor}$$

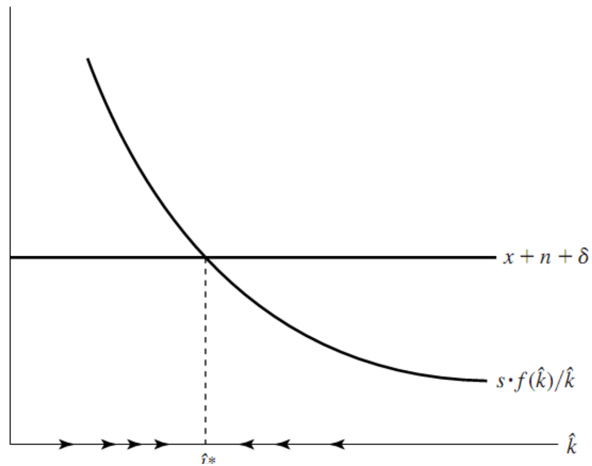
– Then $\hat{y} = \dfrac{Y}{LA(t)} \;\Rightarrow\; \hat{y} = F(\hat{k}, 1) = f(\hat{k})$

Fundamental Equation and Transitional Dynamics

$$\hat{k}^{\cdot} = (\frac{\dot{K}}{LA}) = \frac{\dot{K}(LA) - K(\dot{L}A + \dot{A}L)}{(LA)^2} = \frac{\dot{K}}{LA} - (n + x)$$

$$\Rightarrow \hat{k}^{\cdot} = sf(\hat{k}) - (\delta + n + x)\hat{k} \qquad \text{and at the steady state: } sf(\hat{k}) = (\delta + n + x)\hat{k}$$

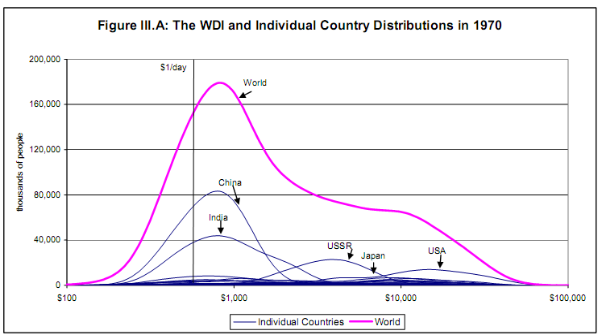
$$\Rightarrow \hat{k}^{\cdot} = sf(\hat{k}) - (\delta + n + x)\hat{k}$$

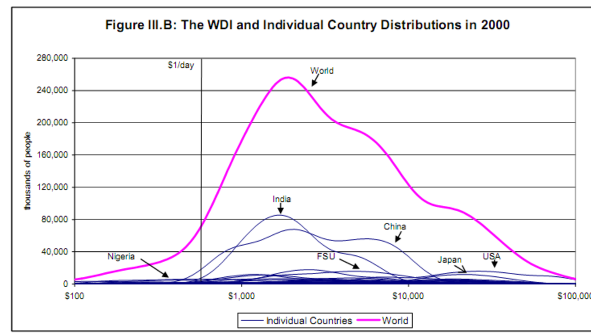


- In the steady state, the variables with hats; $\hat{k}$, $\hat{y}$, $\hat{c}$ are constant. Therefore, the per capita variables; k, y, c grow at the exogenous rate of technological progress, x, and they are on their alanced growth path. The level variables; K, Y, C also grow, and at the rate of $n + x$.

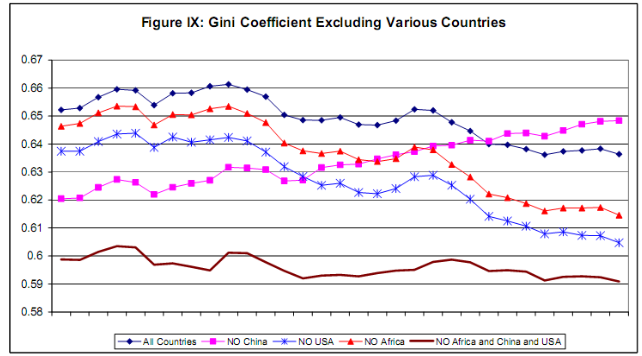
A Note on World Income Distribution

- Two seemingly puzzling facts
  - World Income Inequality, found by integrating individuals' income distribution over large sample of countries, declined between 1970s and 2000s
  - Within country income inequalities has increased over the same period
    - ∗ Explanation: Some of the poorest and most populated countries in the World (most notably China and India, but also many other countries in Asia) rapidly *converged* to the incomes of OECD citizens





Similar picture can be seen from



APPENDIX: Alternative Environment (Markets for Capital and Labor)

- In the Solow-Swan model we assumed that both the labor and capital are fully employed in the production process. In this section we discuss that even if there are markets for labor and capital, i.e. they are supplied by consumers, and demanded by producers (which determine their prices as well), then under the assumption of competitive markets we would obtain the same fundamental dynamic equation for capital stock with the Solow-Swan model.

  – Note 1: In this appendix the capital is owned by consumers, not by firms. This is an innocent assumption that can be justified with several arguments. One of such arguments would be that in reality households own share of stock of firms, and so they own the capital actually. Actually even if the firm owns the capital, the yearly share of the total money that the firm paid on the capital needs to be equal to the rental price of capital. If it is not, say rental price is higher, than some firms buy capital and rent it to some other firms. Eventually this arbitrage condition is explored and two price are equalized. So by using its own capital instead of renting it, firms actually rent their own capital

– Note 2: Even though we use markets and solve for profit maximization of firms in this appendix, we will keep the fixed saving rate assumption of the Solow-Swan model (as if there is a social planner who can decide for the saving rate among other things). When we solve Ramsey model, we will solve for the optimal saving decisions of consumers as well.

Households

- Households hold assets that represents their total wealth, and the change in total assets in the economy is as follows

$$As\dot{s}ets = \omega_t L_t + r_t Assets_t - C_t$$

Assets deliver a rate of return $r(t)$, and labor is paid the wage rate $w(t)$. We assume that each agent inelastically supplies one unit of labor and total labor force in the economy is $L(t)$. The total income received by households is, therefore, the sum of asset and labor income, $r(t) * (assets) + w(t) * L(t)$. Part of this income that is not consumed is accumulated

- In per capita terms ($a_t = \dfrac{assets}{L_t}$) the above equation can be written in as

$$\dot{a}_t = (\frac{Ass\dot{e}ts}{L_t}) = \frac{Ass\dot{e}tsL_t - \dot{L}_t Assets_t}{L_t^2} = \frac{Ass\dot{e}ts}{L_t} - na_t$$
$$\Rightarrow \dot{a}_t = \omega_t + (r_t - n)a_t - c_t$$

Firms

- The firms' problem is to maximize

$$\pi = F(K, L, A) - RK - wL$$

(where the price of output is normalized to1) In fact, firms maximize the present discounted value of the future profits. However, as they are able to change the amount of capital and labor they employ at any point in time, their problem at each period is a static one (the problem becomes intertemporal when we introduce adjustment costs for capital). The FOCs give

$$\frac{\partial F}{\partial K} = R \qquad \frac{\partial F}{\partial L} = w$$

- *Here is the crucial assumption. When we took the derivative of the profit function with respect to K and L, we assumed that R and w are constants and do not change with K and L. This means firms cannot affect their prices, i.e. the markets are competitive. If markets are not competitive, a powerful firms set a price, either for capital or labor for a normalized output price, and K and R, or w and L would be a function of one other, not constants*

- Finally in the previous lecture we saw that for CRTS function; $F(K_t, L_t, A) = L_t f(k_t, A)$, and MPK and MPL are

$$\frac{\partial F}{\partial K} = f'(k_t, A) \ (= R) \quad \& \quad \frac{\partial F}{\partial L} = f(k_t, A) - f'(k_t, A)k_t \ (= \omega)$$

Equilibrium in All Markets

- Households invest on capital assets so they own the capital. Firms pay R_t to rent this capital from households. In real terms households only obtain $R_t - \delta$ (as their capital depreciates at rate δ). Thus the real return on assets is: $r_t = R_t - \delta$.
  - Thus $r_t = f'(k_t, A) - \delta$,

$$\dot{a}_t = \omega_t + (r - n)a_t - c_t \quad \Rightarrow \quad \dot{a}_t = \omega_t + (f'(k_t, A) - \delta - n)a_t - c_t$$

- In a closed Economy: $a_t = k_t$
  - This is because the total capital stock in economy is formed by savings of households. (In an open economy it can be formed by foreign debt as well.) So domestic borrowing equals to domestic lending in a closed economy, implying

$$\dot{k}_t = f(k_t) - f'(k_t, A_t)k_t + f'(k_t)k_t - (\delta + n)k_t - c_t \qquad \dot{k}_t = f(k_t, A_t) - c_t - (\delta + n)k_t$$

  - If one further imposes the assumption that people save constant fraction of their income, i.e. $c_t = (1 - s)f(k_t)$, we have $\dot{k}_t = sf(k_t, A_t) - (\delta + n)k_t$

B- MODELS OF ENDOGENOUS GROWTH

- Endogenous growth means: The balanced growth rate is affected by choices
- In these models determination of long-run growth occurs within the model, rather than by some exogenously growing variables like unexplained technological progress

AK Model (The most famous one)

- Shows that elimination of diminishing returns can lead to endogenous growth

$$Y = AK \qquad A > 0 = constant$$

if we write the equation in terms of y and k

$$\frac{Y}{L} = A\frac{K}{L} \qquad y = f(k) = Ak$$

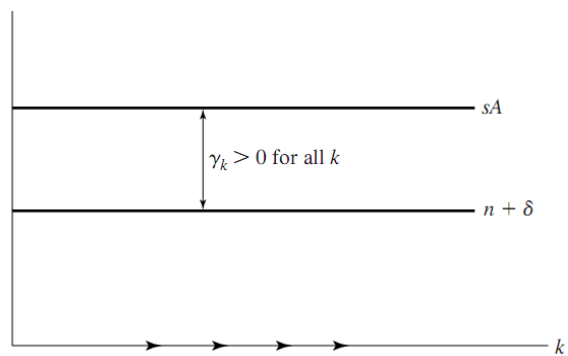
so that average and marginal products are constant. To find $\gamma_k = \frac{\dot{k}}{k}$, let's start from $\dot{k}$

$$\dot{k} = (\frac{\dot{K}}{L}) = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{sAK}{L} - \delta k - nk = sAk - (\delta + n)k$$

and

$$\gamma_k = \frac{\dot{k}}{k} = sA - (\delta + n)$$

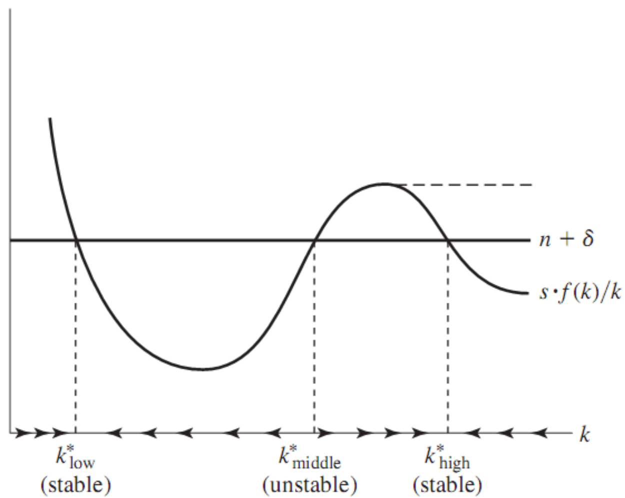
Hence, as long as $sA > (\delta + n)$, sustained growth occurs

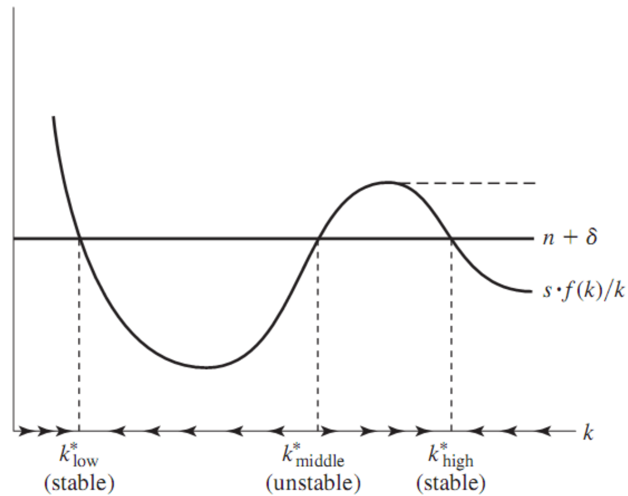


- Notice that $\dot{A}/A = 0$; i.e. growth can occur in the long-run even without an exogenous technological change
- Moreover; $y = Ak \Rightarrow \gamma_{k^*} = \gamma_y$ and $c = (1-s)y \Rightarrow \gamma_{k^*} = \gamma_y = \gamma_c$
- Notes:
  - Change in the parameters (s, A, n, δ) have permanent effect on per capita GDP growth
  - Absence of diminishing return capture both human and physical capital
  - According to the model, independent of the current status of countries (whether they are developed or not), all similar countries (i.e. the countries having similar prameters) grow at the same rate
  - Hence, the model fail to predict absolute or conditional convergence ($\partial\gamma_y/\partial y = 0$)

Growth Models with Poverty Trap

Poverty trap is a stable steady state with low levels of per capita output and capital stock. This outcome is a trap because if agents attempt to break out of it, the economy still has a tendency to return to the low-level steady state. There can be Solow-Swan type or endogenous type of growth models consistent with the idea





- Explanation for the shape of the figure: At low level of development, economies tend to focus on agriculture, a sector in which diminishing return prevail. As an economy develops, it typically concentrates more on industry, services and sectors that involve range of increasing return. Eventually, these benefits may be exhausted and economy again encounters diminishing returns

- Question: How to escape from poverty trap?
  - If there is domestic policy with an increase in s such that $sf(k)/k$ lies above $\delta + n$ at low levels of k
    - Even the temporary increase in saving rate that is maintained long enough to put $k > k_{middle}$, then even if s is lowered back to its initial state, then the economy would not revert back to poverty trap
    - If saving is not enough to satisfy this, then the economy will attain relatively higher steady-state level of per capital stock and income but does not break away poverty trap
  - If country receives donation that is large enough to put the capital above k_{middle}
  - If economy's temporarily high ratio of domestic investment to GDP is financed by international loans, rather than from domestic saving
  - A reduction in population growth rate, n, lowers $\delta + n$ line low enough so that it no longer intersects the $sf(k)/k$ at a low level of k, then economy would escape from the trap