C - RAMSEY MODEL

Households

- Infinitely lived agents
  - Akin to finite lived individuals that are connected through altruism
- *Assumption:* Population grows at a constant rate, $\dfrac{\dot{L}}{L} = n$ i.e. $L_t = L_0 e^{nt}$
- $\max U = \int\limits_0^{\infty} u[c(t)]e^{nt}e^{-\rho t}dt$ where $\rho > 0$ and $c(t) = \dfrac{C(t)}{L(t)}$
  - $e^{-\rho t}$: rate of time preference and $\rho > 0$, meaning that goods are utilized less the later they are received
  - $\rho > n$: U is bounded if c is constant over time
  - *Assumption:* Discount rate within a person's lifetime is the same as that across generations

- It is a closed economy
- Households hold assets which represents their total wealth, and the change in total assets in the economy is as follows

$$\dot{Assets} = \omega_t L_t + r_t Assets_t - C_t$$

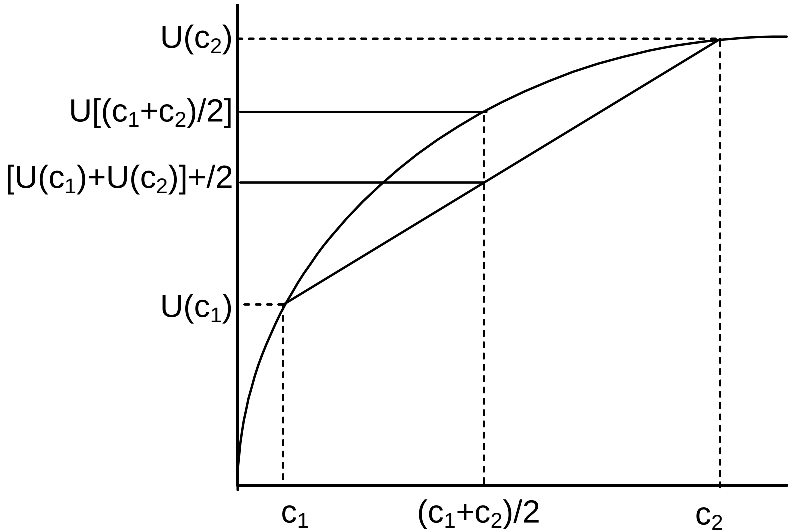
Assets deliver a rate of return $r(t)$, and labor is paid the wage rate $w(t)$. We assume that each agent supplies one unit of labor and total labor force in the economy is $L(t)$. The total income received by households is, therefore, the sum of asset and labor income, $r(t) * Assets + w(t) * L(t)$. The above equation indicates that part of this income that is not consumed is accumulated

- In per capita terms, $a_t = \dfrac{Assets}{L_t}$ is household's net asset (wealth) and the above equation can be written in terms of $\dot{a}_t$

$$A s \dot{s} e t s = \omega_t L_t + r_t Assets_t - C_t$$
$$\dot{a}_t = (\frac{As\dot{s}ets}{L_t}) = \frac{As\dot{s}etsL_t - \dot{L}_t Assets_t}{L_t^2} = \frac{As\dot{s}ets}{L_t} - na_t \quad (1)$$
$$\Rightarrow \dot{a}_t = \omega_t + (r_t - n)a_t - c_t$$

- Households hold assets in the form of ownership claims on capital or as loans, where negative loans represents debts
- The utility function satisfies
$$u'(c) > 0 \qquad u''(c) < 0$$
$$\lim_{c \to 0} u'(c) = \infty \qquad \lim_{c \to \infty} u'(c) = 0$$
  - i.e. the utility households receive from an extra unit of consumption is always positive, but decreases with an increase in c (just like the neoclassical production function does in its inputs)
  - This utility function corresponds to households' desire to smooth their consumption pattern. This comes with its concaveness ($u''(c) < 0$). Instead of consuming c_1 and c_2 separately and having the average utility of $[U(c_1) +$

$U(c_2)]/2$, households prefer to consume $(c_1 + c_2)/2$ in both time periods



Household's Problem: Households try to find the optimal consumption amount at each point in time

$$\max \int_0^\infty u(c(t))e^{-(\rho-n)t}dt$$

$$s.t. \;\; \dot{a}_t = \omega_t + r_t a_t - c_t - n a_t$$

(Control variable: c, State variable: a)

A note on the Solution of the Above Problem

- Households' optimization problem determines their consumption, and part of their income that is not consumed determines the aggregate saving in the economy. Notice that the decision for consumption today affects future consumption via wealth of households, a. (Here c is the *control variable* and a is the *state variable*.) This problem is called *Dynamic Optimization in Continuos Time*. And we cannot solve this continuous time problem just by taking FOCs

- The details of the solution is in Appendix A3 of: Economic Growth: Barro and Sala-i Martin

- In short, first we use a constraint coming from credit markets:

$$\lim_{t\to\infty} a(t)\exp(-\int_0^t [r(v-n]dv) \geqslant 0$$

This is called *Complementary-Slackness condition*, and means that, in the long run a household's debt per person (negative values of $a(t)$) cannot grow as fast as $r(t)-n$, so that debt levels cannot grow as fast as $r(t)$. It rules out Ponzi schemes for debt (If each household can borrow an unlimited amount at the going interest rate, $r(t)$, s/he has an incentive to borrow for financing current consumption and then use future borrowings to roll over the principal and pay all the interest. In this case, the household's debt grows forever at the rate of interest, $r(t)$. This means today's added consumption is effectively free.)

- Secondly, we write the problem as *Hamiltonian* (on the following slide), and take the FOCs accordingly. When we do that, the above constraint is written in the form of

$$\lim_{t\to\infty} \gamma(t)a(t) = 0$$

- This is called *Transversality Condition*
  - $\gamma(t)$ is the (present) value of $a(t)$ at time t. It says that as t goes to infinity, either $a(t)$ should approaches to 0, or if $a(t)$ is positive, then its price, $\gamma(t)$, should be 0 (so that it is harmless to left any positive asset)

Household's Problem (Again):

- The Hamiltonian is

$$\mathtt{H} = u(c)e^{(n-\rho)t} + \lambda(\omega_t + r_t a_t - c_t - n a_t)$$

Notice that the first term on the RHS (right hand side) is the utility and the term inside the parenthesis is the change in income. As a result, in this form λ is the value of an extra unit of asset at time t in units of utility at time 0, and called *shadow price of income*

- Taking FOCs

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-(\rho-n)t} = \lambda \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial a} = -\dot{\lambda} \quad \Rightarrow \quad (r-n)\lambda = -\dot{\lambda} \tag{2}$$

- The Transversality Condition:

$$TVC \quad \Rightarrow \quad \lim_{t\to\infty} \gamma(t)a(t) = 0 \tag{3}$$

Taking derivative of (1) with respect to time

$$(1) \quad \Rightarrow \quad u''(c)\dot{c}e^{(n-\rho)t} - u'(c)(\rho - n)e^{-(\rho-n)t} = \dot{\lambda}$$

substitute this back into (2)

$$(r-n)u'(c)e^{-(\rho-n)t} = u''(c)\dot{c}e^{-(\rho-n)t} - u'(c)(\rho-n)e^{-(\rho-n)t}$$

which results in famous *Euler Equation* (where $u''(c) < 0$):

$$\Rightarrow \quad \frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho) \tag{4}$$

- – When $r = \rho$, the interest rate is equal to future discount rate, households would select a flat consumption profile with $(\frac{\dot{c}}{c} = 0)$
  – If $r > \rho$, households give up consumption today for more consumption tomorrow $(\frac{\dot{c}}{c} > 0)$
  – The more $-\frac{u''(c)c}{u'(c)}$, the less $\frac{\dot{c}}{c}$ responds to an increase in $r > \rho$

- Note: The term $-\dfrac{u''(c)c}{u'(c)}$ is the elasticity of $u'(c)$ with respect to c and can be written as

$$\frac{u''(c)c}{u'(c)} = \frac{\frac{\partial u'(c)}{\partial c}c}{u'(c)} = \frac{\frac{\partial u'(c)}{u'(c)}}{\frac{\partial c}{c}}$$

It is called *the coefficient of relative risk aversion*. So more elastic utility function, the more risk averse households are, meaning the less they would be willing to change their consumption pattern over their life time. Indeed, this term is also *1/elasticity of intertemporal substitution*. This means that the more elastic is the marginal utility, the less elastic is the intertemporal substitution

Ex.

- Assume the functional form $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta} \quad \theta > 0$

  $$u'(c) = (1-\theta)\frac{c^{-\theta}}{1-\theta} = c^{-\theta} \quad u''(c) = -\theta c^{-\theta-1} \quad \Rightarrow \quad \frac{u''(c)c}{u'(c)} = \rho - \frac{-\theta c^{-\theta-1} c}{c^{-\theta}} = -\theta$$

  this utility function has constant relative risk aversion (CRRA), which is θ, and constant intertemporal elasticity of substitution, which is $1/\theta$

  - $\theta \uparrow \ \Rightarrow$ less willing households are to accept deviations from a uniform c
  - $\theta \to 0 \ \Rightarrow u(c)$ approaches a linear form in c (to see this use the l'hopital rule), which makes households indifferent to timing of consumption if $r = \rho$
  - $\theta \to 1 \ \Rightarrow u(c)$ approaches a log-utility form, which we will analyze later

- With this form of utility function: $$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) \tag{5}$$

  - If $\theta \uparrow$, then $\dfrac{\dot{c}}{c}$ responds less to the gap between r & ρ

Transversality Condition

- Remember the Transversality Condition can be written in a way of

$$\boxed{\lim_{t\to\infty} a(t)\exp(-\int_0^t [r(v)-n]dv) = 0} \tag{6}$$

notice that this equation can be derived from the equations (2) & (3)

- With the equality sign this equation tells us that it would be suboptimal for households to accumulate positive assets forever at the rate r or higher because utility would increase if these assets were instead consumed in finite time

Consumption Function

- We have found that

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) \tag{5}$$

- This equation only defines consumption path from one period to the next. If we want to define consumption at time t in terms of the consumption at time 0, we can first use $\exp(-\int_0^t r(v)dv)$, which is the present value factor that converts a unit of income at time t to an equivalent unit of income at $t = 0$. Then we can define the average interest rate between 0 and t as

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(v)dv$$

- c can be written as

$$c_t = c_0 \exp\{\frac{1}{\theta}[\bar{r}(t) - \rho]\}t \tag{7}$$

- Our next question is: 'What is c_0?'
  - We have derived the consumption at any point in time w.r.t. consumption at time 0; hence, we can calculate the total consumption at time 0 (in finite time case it is called life time consumption). Then we can equate the total life time consumption in terms of c_0 to the total (life time) wealth at time 0, and solve for the level of consumption. This is called (life time) *Budget*

Constraint and can be written in a form of

$$\int_0^\infty c_t e^{-[\bar{r}(t)-n]t} dt = a(0) + \int_0^\infty \omega_t e^{-[\bar{r}(t)-n]t} dt \tag{8}$$

This equation says that total (life time) consumption (when it is brought to time 0) must be equal to initial wealth plus total (life time) labor income

– Plugging equation (7) into the equation (8) finds

$$c_0 = ([\int_0^\infty \exp[\frac{1-\theta}{\theta}\bar{r}(t) - \frac{\rho}{\theta} + n]dt) \cdot [a(0) + \int_0^\infty \omega_t e^{-[\bar{r}(t)-n]t} dt] \tag{9}$$

– This equation shows that (depending on θ) an increase in r may increase or decrease c_0. This is because r has two effects. One is what we see in equation (5). It changes consumption relatively between periods, which is called *substitution effect*. Due to this effect an increase in r leads to decline in c_0. The second effect is the *income effect* that we can realize from (life time) Budget Constraint. Due to this effect an increase in r leads to an increase in life time wealth (as it leads to future value of $a(0)$ to increase) and raises c_t at all dates. In fact there is also a third, *wealth effect*, as a result of which an increase in r decreases the current value of future labor income gains (the last term in equation (9)), but this just reinforces the substitution effect for c_0)

* If $\theta < 1$, (small θ), consumers care relatively little about consumption smoothing, and substitution (intertemporal effect) is large
* If $\theta > 1$, substitution effect is relatively weak compared to income effect
* If $\theta = 1$, which is the case with log utility, the two effects exactly cancel out each other

Firms

- We use a neoclassical production function with labor augmenting technology

$$Y(t) = [K(t), L(t) \cdot T(t)]$$

  where technology that grows at a rate of x, i.e. $T(t) = e^{Xt}$ where $T(0) = 1$

- Define effective labor as $\hat{L} = L \cdot T(t)$, so that $Y = F(K, \hat{L})$
- Define variables in terms of per unit of effective labor: $\hat{y} = \dfrac{Y}{LT}$ and $\hat{k} = \dfrac{K}{LT}$
- The production function takes the form $\hat{y} = f(\hat{k})$
  - $R(t)$: rental rate of a unit capital
  - $r(t)$: net rate of return on capital:
  - $r(t) = R(t) - \delta$ = return on bonds (since bonds and capital are perfectly substitutable)

- The profits at any point in time

$$\pi = F(K, \hat{L}) - (r + \delta)K - \omega L = \hat{L} f(\hat{k}) - (r + \delta)K - \omega L$$

  - Notice that although firm's decision is over infinite horizon, this is not a dynamic optimization problem since there is no state variable and firms can rent optimal capital and labor at any point in time. Put it differently, maximizing present value of future profits reduce to a problem of maximizing profit in each period
    - The situation is different with the adjustment cost of capital

- Taking r and ω as given (in the competitive environment) FOCs find MPK and MPL:

$$\frac{\partial \pi}{\partial K} = 0 \implies 0 = \frac{\partial(\hat{L}f(\hat{k}))}{\partial K} - (r+\delta) \implies 0 = \frac{\partial(\hat{L}f(\hat{k}))}{\partial \hat{k}}\frac{\partial \hat{k}}{\partial K} - (r+\delta)$$

$$\implies 0 = \hat{L}f'(\hat{k})\frac{1}{LT} - (r+\delta) \implies r = f'(\hat{k}) - \delta \quad (10)$$

$$\frac{\partial \pi}{\partial L} = 0 \implies 0 = \frac{\partial(\hat{L}f(\hat{k}))}{\partial L} - \omega \implies 0 = Tf(\hat{k}) + \hat{L}\frac{\partial(f(\hat{k}))}{\partial \hat{k}}\frac{\partial \hat{k}}{\partial L} - \omega$$

$$0 = Tf(\hat{k}) - \hat{L}f'(\hat{k})\frac{K}{L^2T} - \omega \implies 0 = Tf(\hat{k}) - Tf'(\hat{k})\hat{k} - \omega$$

$$\text{remember that } T(t) = e^{Xt} \implies \omega = [f(\hat{k}) - f'(\hat{k})\hat{k}]e^{xt} \quad (11)$$

- *Checking for the profits:* Plugging ω and r into the profit function, we have

$$\pi = \hat{L}[f(\hat{k}) - (r+\delta)\hat{k} - \omega e^{-xt}] = \hat{L}[f(\hat{k}) - f'(\hat{k})\hat{k} - f(\hat{k}) + f'(\hat{k})\hat{k} = 0$$

  – so firms have zero profits $\forall t$

Equilibrium

- Closed economy $\Rightarrow \quad a = k$
- The growth rate of capital per effective worker

$$\hat{k} = \frac{k}{T} = ke^{-xt} \quad \Rightarrow \quad \hat{k}^{\cdot} = k^{\cdot}e^{-xt} - kxe^{-xt} = k^{\cdot}e^{-xt} - \hat{k}x$$

- The equation, $\dot{a} = \omega + ra - c - na$, can be written as

$$\dot{k} = \omega + rk - c - nk$$

  Then we can use the relation between $\hat{k}^{\cdot}$ and $\dot{k}$

$$\hat{k}^{\cdot} = (\omega + rk - c - nk)e^{-xt} - \hat{k}x$$

- this equation, together with (10) & (11)can be used to derive

$$\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k} \qquad (12)$$

  where $\hat{c} = \frac{c}{T} = ce^{-xt}$

- Equation (12) is the resource constraint of the economy.
- Since $\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$, it follows that

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{c}}{c} - x = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x] \tag{13}$$

- Together with equations (12) and (13), the TVC in equation (6) implies that

$$\lim_{t\to\infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x] dv\} = 0 \tag{14}$$

  - TVC's explanation: $f'(\hat{k}) - \delta > n + x$

    The steady state rate of return on capital is higher than steady state growth of K

- The system of equations (12) and (13), the initial condition of $\hat{k}(0)$, and TVC determine the time paths $\hat{c}$ and $\hat{k}$

Steady State

- Steady state growth rates of $\hat{k}$ and $\hat{c}$ are zero. It can be shown that the level of variables, K, C and Y grow at the rate $x + n$

Transitional Dynamics

- (12): $\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x] \quad \Rightarrow \quad \dot{\hat{c}} = 0$ if and only if $\hat{c} = 0$ or $f'(\hat{k}) = \delta + \rho + \theta x$

  So at the steady state with positive level of consumption:

$$f'(\hat{k}^*) = \delta + \rho + \theta x$$

- (13): $\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (n+\delta+x)\hat{k} \quad \Rightarrow \quad \dot{\hat{k}} = 0$ if and only if $\hat{c} = f(\hat{k}) - (n+\delta+x)\hat{k}$

  There are many k values satisfying this equation. But it is maximized at the golden rule level of capital: $\max \hat{c} = \max f(\hat{k}) - (n + \delta + x)\hat{k}$

$$\Rightarrow \quad f'(\hat{k}_{gold}) = n + \delta + x$$

- (14): $\lim_{t\to\infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x] dv\} = 0$. If capital converges to steady state, then TVC requires the steady-state rate of return, $f'(\hat{k}^*) - \delta$, to exceed $x + n$, the steady-state growth rate of K

$$f'(\hat{k}^*) - \delta > n + x$$

- Given these equations, we can compare the steady state of capital, $\hat{k}^*$, and $\hat{k}_{gold}$

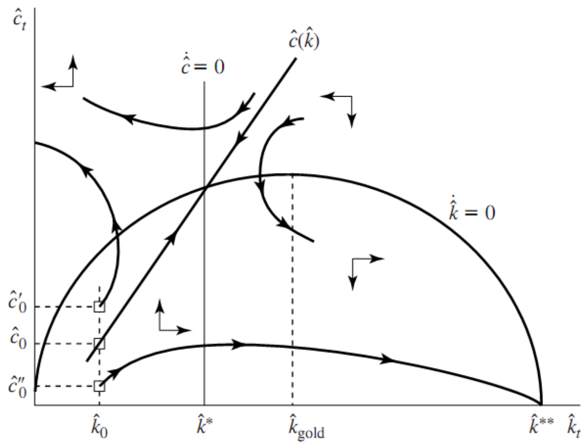
$$f'(\hat{k}^*) - \delta > n + x \; [= f'(\hat{k}_{gold}) - \delta] \quad \Rightarrow \quad \hat{k}^* < \hat{k}_{gold}$$

- Results:

  1. Inefficient oversaving ($\hat{k}^* > \hat{k}_{gold}$) cannot occur in the optimizing framework compared to the Solow-Swan model with a constant saving rate
  2. The optimizing households does not save enough to attain k^{gold}. The reason is the impatience reflected in the effective discount rate, $\rho + \theta x$, makes it not worthwhile to sacrifice more of current consumption to reach the maximum amount
     - The θx part of the effective discount rate picks up the effect from diminishing marginal utility of consumption due to growth of c at the rate x
  3. Parameters that describe production function or ρ and θ affect long run level of variables, but do not affect the steady-state growth rates

Phase Diagram

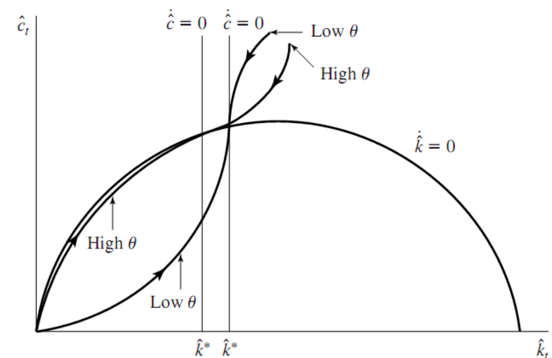
- (12): $\frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x]$ [$\hat{c}$ is rising for $\hat{k} < \hat{k}^*$ (so the arrows point upward in this region) and falling for $\hat{k} > \hat{k}^*$]
- (13): $\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k}$ [$\hat{k}$ is falling for values of $\hat{k}$ above the solid curve and rising for values of $\hat{k}$ below the curve]



The $\hat{c}^{\cdot} = 0$ and the $\hat{k}^{\cdot} = 0$ lines cross three times. So there are three steady states (the first one is the origin$-\hat{c} = 0$ and $\hat{k} = 0-$, the second steady state corresponds to $\hat{k}^*$ and $\hat{c}^*$, and the third one involves a positive capital stock$-\hat{k}^{**} > 0$). For any initial positive capital stock, $k(0)$ the only stable equilibria is the one that stays on $\hat{c}(\hat{k})$ locus that reaches the second steady state, $\hat{k}^*$ and $\hat{c}^*$

- The stable arm, $\hat{c}(\hat{k})$, is a *policy function* that expresses the equilibrium $\hat{c}$, the control variable, as a function of $\hat{k}$, the state variable
- Suppose $\hat{k}(0) < \hat{k}^*$ (i.e. in the future, capital and consumption will be higher)
  - $\theta \uparrow \Rightarrow$ Households have a strong preference for smoothing consumption; They shift consumption from the future to the present; Low rate of investment and thus slower convergence; Stable arm lies close to $\hat{k}^{\cdot} = 0$ schedule
  - $\theta \downarrow \Rightarrow$ Households more willing to postpone c for high rates of return; Stable arm is closer to horizontal axis; Faster transition to the s.s.

The Shape of the Stable Arm:



Behavior of the Saving Rate

- The gross saving rate

$$s = 1 - \frac{\hat{c}}{f(\hat{k})}$$

The behavior of the saving rate is ambiguous because of the offsetting impacts from substitution and income effects

  - *Substitution Effect:* As country gets richer: $\hat{k} \uparrow \;\Rightarrow\; f'(\hat{k}) \downarrow \Rightarrow r \downarrow \Rightarrow$ An intertemporal substitution reduces, current consumption increases, saving declines
  - *Income Effect:* Income per effective worker in a poor economy is far below the long-run income of the economy. Since households like to smooth consumption, this means they consume higher fraction of their income when they are poor, so saving would be low when $\hat{k}$ is low. As $\hat{k}$ rises, gap between current and permanent income diminishes; hence, consumption falls in relation to current income and saving increases

APPENDIX 1: Alternative Environment

- Separation of households and firms is not crucial to the analysis we have carried out. The same results would be obtained if one allows households to perform the functions of the firms by employing adult family members as workers

Benevolent Social Planner

- Maximizing utility of the consumer s.t. the resource constraint of the economy

$$\max \int_0^\infty u(c(t))e^{-(\rho-n)t}dt \qquad s.t. \quad \hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n+\delta+x)\hat{k}$$

- Having a state variable, $\hat{k}$, this equation can be solved by Hamiltonian

$$\mathtt{H} = e^{-(\rho-n)t}u(c) + \lambda(f(\hat{k}) - ce^{-xt} - (n+\delta+x)\hat{k})$$

- The first order conditions

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-(\rho-n)t} - \lambda e^{-xt} = 0 \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial \hat{k}} = -\dot{\lambda} \quad \Rightarrow \quad -\dot{\lambda} = \lambda[f'(\hat{k}) - (n+\delta+x)] \tag{2}$$

- Taking first ln and then derivative of (1), solving for $\hat{c}$, and combining it with (2) leads to

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x],$$

which is equivalent to (13). The solution for the planner will therefore be the same as that for the decentralized economy. Since a benevolent social planner with dictatorial powers will attain a Pareto optimum, the results for the decentralized economy—which coincide with those of the planner—must also be Pareto optimal.

APPENDIX 2: Dynamic Optimization in Discrete Time (Dynamic Programming Approach)

- Social planner maximizes the (expected) utility function of the representative consumer

$$\max_{c_t} U_t = \max_{c_t} E_0 \sum_{t=0}^{\infty} (\frac{1}{1+\rho})^t u(c_t)$$

where $\beta \in (0,1)$ and the flow utility or one-period utility function is strictly increasing ($u\prime(c) > 0$) and strictly concave ($u''(c) < 0$), subject to the resource constraint of the economy

$$c_t + k_{t+1} \leq y_t + (1-\delta)k_t$$

and the restriction that the capital stock cannot fall below zero.

$$k_t \geq 0 \quad \forall t$$

- Now we will solve consumption in terms of existing capital stock: $c_t(k_t)$. In the Dynamic Programming problem k_t is called as the *state variable* (showing the state, well being, of society), and c_t is the *control variable.* Finally $c_t(k_t)$ is called the *policy function*

- Consumers get utility $u(\cdot)$ from consuming c. But since they are able to afford c as they have capital, k, there should be a function, that we call $V(\cdot)$, that would give correspondence of k in terms of utility. Hence the consumer problem at the final period T, instead of $u(c_T(k_T))$, can be written as $V_T(k_T)$, which is called the *Value Function*

- As a result, the consumer problem in period $T-1$ can be written as

$$\max_{c_{T-1}}(u(c_{T-1}) + \frac{1}{1+\rho}V_T(k_T))$$

given k_{T-1}, choosing c_{T-1} determines k_T, which determines the utility for the rest of the periods

- The same procedure can be applied to earlier periods *recursively (backward recursion)*. In general, the problem can be written in terms of *the Bellman equation*:

$$V_t(k_t) = \max_{c_t}(u(c_t) + \frac{1}{1+\rho}V_{t+1}(k_{t+1}))$$

subject to

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

where

$$V(k_0) = \sum_{t=0}^{\infty}(\frac{1}{1+\rho})^t u(c_t^*(k_0))$$

where $c_t^*(k_0)$ is the optimal consumption sequence for each possible value of k_0

  – FOC w.r.t. c_t gives

$$u'(c_t) = \frac{1}{1+\rho}V'_{t+1}(k_{t+1})$$

  – FOC w.r.t. k_t gives

$$V'_t(k_t) = \frac{1+f'(k_t)-\delta}{1+\rho}V'_{t+1}(k_{t+1})$$

  * Actually there are two additional terms coming from the derivative of c_t w.r.t. k_t, but they cancel out due to the first FOC

- Combining FOCs

$$V_t'(k_t) = (1 + f'(k_t) - \delta)u'(c_t)$$

- Inserting this equation into the first FOC, we find the Euler Equation

$$u'(c_t) = \frac{1 + f'(k_{t+1}) - \delta}{1 + \rho} u'(c_{t+1})$$

  and notice that in the market solution $f'(k_{t+1}) - \delta = r$. In that case the final equation reduces to

$$u'(c_t) = \frac{1 + r}{1 + \rho} u'(c_{t+1})$$

- To see that the final (Euler) equation is analogous to its continuos time version, use the Taylor Approximation:

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x + ..$$

- Using the first order approximation for $u'(c_{t+1})$ in the discrete time Euler Equation

$$u'(c_t) = \frac{1+r}{1+\rho}(u'(c_t) + u''(c_t)(c_{t+1} - c_t))$$

which can be simplified as

$$\frac{c_{t+1} - c_t}{c_t} = -\frac{u'(c_t)}{u''(c_t)c_t}(\frac{r-\rho}{1+\rho})$$

which is, given that $1+\rho \simeq 1$, identical to continuos time version of the Euler Equation, repeated below

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r-\rho) \tag{5}$$

APPENDIX 3: Utility Functions

- Relative Risk Aversion (RRA)= $-\dfrac{\dfrac{\partial u^{'}(c)}{u^{'}(c)}}{\dfrac{\partial c}{c}} = -\dfrac{u''(c)c}{u'(c)}$

- Absolute Risk Aversion (ARA)= $-\dfrac{\dfrac{\partial u^{'}(c)}{u^{'}(c)}}{\partial c} = -\dfrac{u''(c)}{u'(c)}$

  – If the coefficient of relative risk aversion (RRA) is constant for a utility function, then we say it is CRRA type. If coefficient of absolute risk aversion (ARA) is constant, then we say it is CARA type

- Assume the functional form: $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta} \quad \theta > 0$

$$u'(c) = (1-\theta)\frac{c^{-\theta}}{1-\theta} = c^{-\theta} \qquad\qquad u''(c) = -\theta c^{-\theta-1}$$

$$-\frac{u''(c)c}{u'(c)} = \theta \qquad\qquad -\frac{u''(c)}{u'(c)} = \frac{\theta}{c}$$

$\Rightarrow$ The isoelastic utility function has CRRA form

Also $u''(c) = -\theta c^{-\theta-1} < 0$ and $u'''(c) = \theta(1+\theta)c^{-\theta-2} > 0$

- Assume the functional form: $u(c) = \log c$

$$u'(c) = \frac{1}{c} \qquad u''(c) = -\frac{1}{c^2}$$

$$-\frac{u''(c)c}{u'(c)} = 1 \qquad -\frac{u''(c)}{u'(c)} = \frac{1}{c}$$

$\Rightarrow$ The logarithmic utility function has CRRA form

Also $u''(c) = -\frac{1}{c^2} < 0$ and $u'''(c) = \frac{2}{c^3} > 0$

The logarithmic utility function is special isoelastic utility function with $\theta = 1$

- Assume the functional form: $u(c) = -(1/\theta)e^{-\theta c} \qquad \theta > 0$ and $\theta \neq 0$

$$u^{'}(c) = -(1/\theta)(-\theta)e^{-\theta c} = e^{-\theta c} \qquad u^{''}(c) = -\theta e^{-\theta c}$$

$$-\frac{u''(c)c}{u'(c)} = \theta c \qquad -\frac{u'(c)}{u''(c)} = \theta$$

$\Rightarrow$ Exponential Utility function has CARA form

Also $u''(c) = -\theta e^{-\theta c} < 0$ and $u'''(c) = \theta^2 e^{-\theta c} > 0$

- Assume the functional form (Quadratic utility): $u(c) = c - \frac{b}{2}c^2$

$$u'(c) = 1 - bc > 0 \qquad u''(c) = -b < 0 \qquad u'''(c) = 0$$