**INVESTMENT**

- In the baseline model of investment, which is discussed below, firms face a perfectly elastic supply of capital goods and can adjust their capital stocks costlessly
- Next, the $q$ theory model of investment is analyzed. The model's key assumption is that firms face costs of adjusting their capital stocks



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## Baseline Model

### The Desired Capital Stock

- $r_K$ is the rental price of capital
- $K$ is the amount of capital
- The firm's profits at a point in time are given by

$$\pi(K, X_1, X_2, ..., X_n) - r_K K$$

  where the $X$'s are variables that the firm takes as given, $\pi_K > 0$, $\pi_{KK} < 0$

- The result of the profit maximization problem is

$$\pi_K(K, X_1, X_2, ..., X_n) = r_K \quad (1)$$

  That is, the firm rents capital up to the point where its marginal revenue equals its rental price.

**The User Cost of Capital**

- Suppose the real market price of the capital at time t is $p_K(t)$
- The cost of using the capital can be written as follows:

$$r_K(t) = r(t)p_K(t) + \delta p_K(t) - \dot{p}_K(t) \tag{2}$$

- This is because the capital has three costs to the firm
  - First, the firm forgoes the interest it would receive if it sold the capital and saved the proceeds. This has a real cost of $r(t)p_K(t)$ per unit time, where $r(t)$ is the real interest rate.
  - Second, the capital is depreciating. This has a cost of $\delta p_K(t)$ per unit time, where $\delta$ is the depreciation rate.
  - Third, the price of the capital may be changing. This has a cost of $-\dot{p}_K(t)$ per unit time.
  - When $\delta = 0$, (2) reduces to

$$r_K(t) = r(t)p_K(t) - \dot{p}_K(t) \tag{2*}$$

**What Happens, for example, if r increases?**

- According to (2), if $r$ increases, then $r_K$ increases. Given that

$$\pi_K(K, X_1, X_2, ..., X_n) = r_K \tag{1}$$

- This affects the capital stock in a way that

$$\pi_{KK}(K, X_1, X_2, ..., X_n)\frac{\delta(K, X_1, X_2, ..., X_n)}{\delta r_K} = 1$$

- Since $\pi_{KK}$ is negative, this equations implies that $K$ is decreasing in $r_K$

**Problems with the Baseline Model**

- Suppose that the Federal Reserve reduces interest rates
- This reduces the cost of capital, $r_K$, and increases the capital stock
- Meaning that a discrete change in the capital stock requires an infinite rate of investment
- For the economy as a whole, however, investment is limited by the economy's output; thus aggregate investment cannot be infinite
- The second problem with the model is that it does not identify any mechanism through which expectations affect investment decisions.

## A Model of Investment with Adjustment Costs

- To address the problems mentioned above, the standard theory uses the presence of costs to changing the capital stock.
  - Internal adjustment costs arise when firms face direct costs of changing their capital stocks
  - External adjustment costs arise when each firm faces a perfectly elastic supply of capital but where the price of capital goods relative to other goods adjusts so that firms do not wish to invest or disinvest at infinite rates
- In what follows the adjustment costs are assumed to be internal.
- The model is known as the $q$ theory model of investment.

**Assumptions**

- Consider an industry with $N$ identical firms.
- A representative firm's real profits are proportional to its capital stock, $\kappa_t$, and decreasing in the industry-wide capital stock, $K_t$, thus they take the form

$$\pi(K_t)\kappa_t$$

  where $\pi'(\cdot) < 0$
- The firm's profits are proportional to its capital as the production function has constant returns to scale, output markets are competitive
- Firms face costs of adjusting their capital stocks. The adjustment costs are a convex function of the rate of change of the firm's capital stock, $\dot{\kappa}$. Specifically, the adjustment costs, $C(\dot{\kappa})$, satisfy $C(0) = 0$, $C'(0) = 0$, and $C''(\cdot) > 0$

- These assumptions imply that it is costly for a firm to increase or decrease its capital stock, and that the marginal adjustment cost is increasing in the size of the adjustment
- The purchase price of capital goods is constant and equal to 1; thus there are no external adjustment costs.
- The depreciation rate is assumed to be zero

$$\dot{\kappa}_t = I_t$$

- The firm maximizes the present value of these profits

$$\Pi = \int_{t=0}^{\infty} e^{-rt}[\pi(K_t)\kappa - I_t - C(I_t)]dt,$$

**A Discrete-Time Version of the Firm's Problem**

- In discrete time, the firm's objective function

$$\tilde{\Pi} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [\pi(K_t)\kappa_t - I_t - C(I_t)]$$

and that

$$\kappa_t = \kappa_{t-1} + I_t$$

- The Lagrangian for the firm's maximization problem is

$$\pounds = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [\pi(K_t)\kappa - I_t - C(I_t)] + \sum_{t=0}^{\infty} \lambda_t [\kappa_{t-1} + I_t - \kappa_t]$$

- $\lambda_t$ gives the marginal impact of an exogenous increase in $\kappa_t$ on the lifetime value of the firm's profits discounted to time 0.

- This discussion implies that if we define $q_t = (1+r)^t \lambda_t$, then $q_t$ shows the value to the firm of an additional unit of capital at time $t$ in time-t dollars. With this definition, the Lagrangian can be rewritten as

$$£ = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}[\pi(K_t)\kappa - I_t - C(I_t) + q_t(\kappa_{t-1} + I_t - \kappa_t)]$$

- The first-order condition for the firm's investment in period $t$ is therefore

$$\frac{1}{(1+r)^t}[-1 - C'(I_t) + q_t] = 0$$

which is equivalent to

$$1 + C'(I_t) = q_t$$

The cost of acquiring a unit of capital equals the purchase price (which is fixed at 1) plus the marginal adjustment cost. This equation states that the firm invests to the point where the cost of acquiring capital equals the value of the capital.

- The first-order condition for $\kappa_t$ is

$$\frac{1}{(1+r)^t}[\pi(K_t) - q_t] + \frac{1}{(1+r)^{t+1}}q_{t+1} = 0$$

- Multiplying this expression by $(1+r)^{t+1}$ and rearranging yields

$$(1+r)\pi(K_t) = (1+r)q_t - q_{t+1}$$

- If we define $\Delta q_t = q_{t+1} - q_t$, this equation can be rewritten as

$$\pi(K_t) = \frac{1}{1+r}(rq_t - \Delta q_t) \tag{3}$$

- Intuitively, owning a unit of capital for a period requires forgoing $rq_t$ of real interest and involves changes in the value of capital gains of $q_t$

- This condition is analogous to the condition in the model without adjustment costs (see Equation (2*)) that the firm rents capital to the point where its marginal revenue product equals its rental price and there is no depreciation

- A second way of interpreting (3) is as follows

$$q_t = \pi(K_t) + \frac{1}{1+r} q_{t+1}$$

Hence, $q_t$ equals the amount the capital contributes to the firm's objective function this period, $\pi(K_t)$, plus the value the firm will attach to the capital next period, $q_{t+1}/(1+r)$, so that its valuations in the two periods are inconsistent.

- This equation implies that

$$\begin{aligned} q_0 &= \pi(K_0) + \frac{1}{(1+r)} q_1 \\ &= \pi(K_0) + \frac{1}{(1+r)} [\pi(K_1) + \frac{1}{(1+r)} q_2] \\ &= \ldots \\ &= \lim_{T-\infty} \left\{ \left[ \lim_{T-\infty} \sum_{t=0}^{T-1} \frac{1}{(1+r)^t} \pi(K_t) \right] + \frac{1}{(1+r)^T} q_T \right\} \end{aligned}$$

- There is also the transversality condition

$$\lim_{T-\infty} \frac{1}{(1+r)^T} q_T = 0$$

- $q$ summarizes all information about the future that is relevant to a firm's investment decision. It shows how an additional dollar of capital affects the present value of profits. Thus the firm wants to increase its capital stock if $q$ is high and reduce it if $q$ is low.

- There is another interpretation of $q$. A unit increase in the firm's capital stock increases the present value of the firm's profits by $q$, and thus raises the value of the firm by $q$. Thus $q$ is the market value of a unit of capital.

- The ratio of the market value to the replacement cost of capital (which is assumed 1 here) is known as Tobin's $q$ (Tobin, 1969)

**Some Practical Results:**

- James Tobin suggested that firms' decision to invest depends on the following ratio, called Tobin's $q$

$$q = \frac{\text{Market value of the company's (or the market''s) installed capital}}{\text{Market price of the installed capital or its replacement cost}}$$

- For example, a low $q$ (between 0 and 1) means that the cost to replace a firm's assets is greater than the value of its stock. This implies that the stock is undervalued.

- Conversely, a high $q$ (greater than 1) implies that a firm's stock is more expensive than the replacement cost of its assets, which implies that the stock is overvalued

- An undervalued company, one with a ratio of less than one, would be attractive to corporate raiders or potential purchasers, as they may want to purchase the firm instead of creating a similar company

- When applied to the market as a whole, we can gauge whether an entire is relatively overbought or undervalued

- If Tobin's $q$ is greater than 1.0, then the market value is greater than the value of the company's recorded assets. This suggests that the market value reflects some unmeasured or unrecorded assets of the company. High Tobin's $q$ values encourage companies to invest more in capital because they are "worth" more than the price they paid for them.

- Tobin's $q$ depends on current and future expected benefits from installed capital.If a government announces a tax cut from the beginning of the next year, the expected profits for the owners of capital will be high. This will raise the value of stock immediately, raise Tobin's $q$ and, therefore, encourage business managers to make more investment now

- Tobin's theory posits that investment should be made when the change that it creates in the firm's market value exceeds its cost. The change in market value relative to capital cost is called 'marginal $q$' and can differ from the level of the market-value-to-cost ratio, called 'average $q$', on which data are available for statistical tests.

  – Our analysis implies that what is relevant to investment is marginal $q$. Marginal $q$ is likely to be harder to measure than average $q$—the ratio of the total value of the firm to the replacement cost of its total capital stock.

- Tobin's $q$ is still used in practice, but does not always perform well to predict investment results (e.g. Housing Market)