**TOBB-ETU, Economics Department**
**Macroeconomics II (ECON 532)**
**Practice Problems I (Solutions)**

## Q: The Solow-Swan Model: Constant returns

Prove that, if the production function exhibits constant returns, all firms will use the same proportion of capital to labor.

**Solution:** Profits for firms are:

$$\pi = F(K, AL) - rK - wL$$

which we can rewritten as:

$$\pi = L[F(K/L, A) - rK/L - w]$$

Since $L$ cannot be negative, maximizing the above expression implies maximizing the term inside the bracket. Since there is a unique value of $K/L$ which maximizes that expression for a given set of prices, that means all firms will choose the same $K/L$, no matter how big or small (Notice that prices are determined by the capital/labor ratio as well)


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## Q: The Solow-Swan Model: Shares and Golden Rule

**a-)** Consider a Solow model without a technological growth. Derive the Golden Rule of Capital Accumulation. What conditions (parameters) determine the golden rule capital stock?

**Solution:** Golden Rule of Capital Accumulation is the amount of capital that maximizes consumption at the steady state.

We know that the steady state level of consumption is: $c^* = (1-s)f(k^*)$. The saving rate that maximizes this function can be found by

$$\max_s c^* = \max_s (1-s)f(k^*(s))$$

We have also shown that at the steady state: $sf(k^*) = (\delta + n)k^*$ (the investment is equal to the effective depreciation). Hence, the maximization problem can be written in terms of steady state capital amount

$$\max_{k^*} c^* = \max_{k^*} [f(k^*) - (\delta + n)k^*]$$

Taking the FOC gives the condition for the Golden Rule of Capital Accumulation

$$f'(k_{gold}) = (\delta + n)$$

**b-)** Consider a Cobb Douglas technology with parameter $0 < \alpha < 1$. Imagine that the depreciation rate of capital is $\delta = 0.10$ and the rate of population growth is $n = 0.01$. Compute the golden rule capital stock $k_{gold}$ as a function of $\alpha$. What is the golden rule capital stock when $\alpha = 1/2$?

**Solution:** Using the condition found in part (a)

$$f'(k_{gold}) = (\delta + n)$$

and inserting for parameter values and the production function

$$\alpha {k_{gr}^{*}}^{\alpha-1} = 0.01 + 0.1 = 0.11$$

which results in

$$k_{gr}^{*} = (1/0.22)^2 = (50/11)^2$$

**c-)** Imagine that the saving rate in the economy is the constant $s$ (with $0 < s < 1$). What would be the value of $s$ that delivers $k_{gold}$ as the steady state capital stock?

**Solution:** At each steady state of capital accumulation (either it is golden rule or not) the change in capital stock is 0

$$\dot{k}^*_{gr} = sf(k^*_{gr}) - (n+\delta)k^*_{gr} = 0$$

which gives us the Golden Rule Level of Saving Rate

$$s = \frac{(n+\delta)k^*_{gr}}{{k^*_{gr}}^{\alpha}} = \frac{(n+\delta)}{{k^*_{gr}}^{\alpha-1}}$$

from part (b) we know that $\alpha {k^*_{gr}}^{\alpha-1} = (\delta + n)$. Thus

$$s = \frac{n+\delta}{(n+\delta)/\alpha} = \alpha$$

so the optimal saving rate is equal to the capital share in the production function

**d-)** Suppose now that inputs earn their marginal products. Show that when owners of capital save all their income and workers consume all their income, then the economy reaches the golden rule capital stock

**Solution**: We know from Cobb Douglas technology with competitive markets that inputs earn their marginal product: $Y = F_K K + F_L L$. This equation can also be written in per capita terms: $y = F_K k + F_L$. We have also shown in the lecture that

$$\frac{\partial Y}{\partial K} = f'(k) \quad \text{and} \quad \frac{\partial Y}{\partial L} = f(k) - f'(k)k$$

When owners of capital save all their income and workers consume all their income, the total saving in the economy is

$$k * MPK = \frac{\partial(k^{\alpha})}{\partial k} = k\alpha k^{\alpha-1} = \alpha k^{\alpha} = \alpha y$$

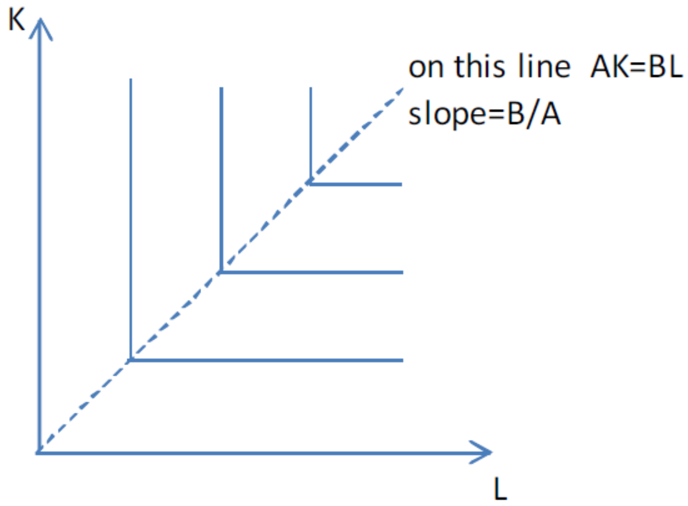
So $\alpha$ fraction of total output is saved. And this is the same amount we found in part (c) that is needed for each golden rule capital stock

## Q: The Solow-Swan Model: Harrod-Domar

Consider the Solow-Swan model with constant savings rate, s. Imagine that the technology is Leontief so: $Y = \min(AK, BL) \quad (1)$

**a-)** Draw indifference curves in the $K - L$ space

**Solution**



**b-)** Write down $y = Y/L$ as a function of $k$

**Solution**

$$Y = \min(AK, BL) = L \min(Ak, B) \quad \Rightarrow \quad y = \begin{matrix} Ak & \text{if } k < B/A \\ B & \text{if } k > B/A \end{matrix}$$

**c-)** If the rate of population growth is zero ($n = 0$), show that the growth rate of capital is as follows:

$$\frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$$

**Solution**

$$\dot{K} = sY - \delta K \quad \Rightarrow \quad \dot{k} = sy - (n + \delta)k \quad \Rightarrow \quad \frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$$

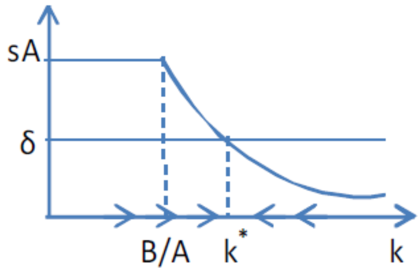
Draw the savings and the depreciation lines as a function of $k$. Find the steady state capital stock, $k^*$, for each of the cases written below. What are the dynamics of $k$ over time? Do you see any problems with those steady states?

**d-)** $sA > \delta$ **e-)** $sA < \delta$ **f-)** $sA = \delta$

**Solution**: We know the dynamic equation for capital: $\frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$

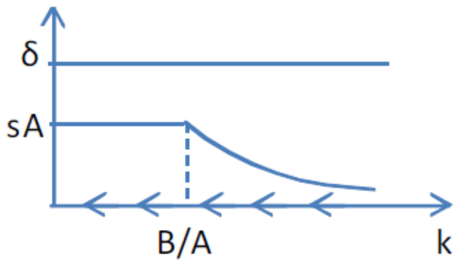
Hence,

$sA > \delta$



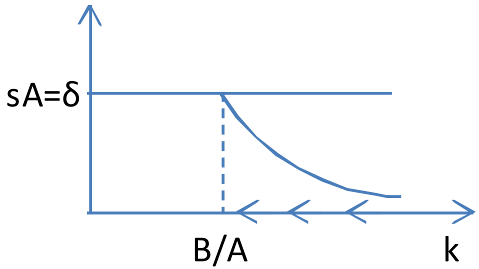

idle machines

$sA < \delta$




idle labor

$sA = \delta$




at $k^* = B/A$
without any idle factor

**Q: The Solow-Swan Model: CES (Constant Elasticity of Substitution) Production Function**

Consider the following production function

$$Y = F(K, L) = A(a[bK]^{\psi} + (1-\alpha)[(1-b)L]^{\psi})^{1/\psi}$$

where $0 < a < 1$, $o < b < 1$ and $\psi < 1$

**a-)** Show that the elasticity of substitution between capital and labor is $\dfrac{1}{1-\psi}$

**Solution:** Suppose that $a[bK]^{\psi}+(1-\alpha)[(1-b)L]^{\psi} = C$. The elasticity of substitution is a measure of the curvature of the isoquants. The slope of an isoquant is (by implicit function theorem)

$$\frac{\partial L}{\partial K_{\text{isoquant}}} = -\frac{\partial F(\cdot)/\partial K}{\partial F(\cdot)/\partial L} == -\frac{AC^{(1-\psi)/\psi}ab^{\psi}K^{\psi-1}}{AC^{(1-\psi)/\psi}(1-a)(1-b)^{\psi}L^{\psi-1}} = -\frac{ab^{\psi}}{(1-a)(1-b)^{\psi}}(\frac{L}{K})^{1-\psi}$$

And, as a result, the elasticity is

$$E = [\frac{\partial(Slope)}{\partial(L/K)}\frac{L/K}{Slope}]^{-1} = [-(1-\psi)\frac{ab^{\psi}}{(1-a)(1-b)^{\psi}}(\frac{L}{K})^{-\psi}\frac{\frac{L}{K}}{-\frac{ab^{\psi}}{(1-a)(1-b)^{\psi}}(\frac{L}{K})^{1-\psi}}]^{-1} = \frac{1}{1-\psi}$$

**b-)** Show that when $\psi \to 0$ (unit elasticity of substitution), the production function approaches the following Cobb Douglas technology

$$Y = cons\tan t \cdot K^{\alpha} L^{1-\alpha}$$

**Solution:**

$$\lim_{\psi \to 0} F(K, L) = \lim_{\psi \to 0} A(a[bK]^{\psi} + (1-a)[(1-b)L]^{\psi})^{1/\psi}$$

taking the ln

$$\lim_{\psi \to 0} [\log(Y)] = \log(A) + [\frac{\log(a[bK]^{\psi} + (1-a)[(1-b)L]^{\psi})}{\psi}]_{\psi \to 0} = \log(A) + \frac{\log(1)}{0} = \log(A) + \frac{0}{0}$$

apply l'Hôpital's rule by taking the derivative of the numerator and denominator

$$\lim_{\psi \to 0} [\log(Y)] = \log(A) + [\frac{a(bK)^{\psi} \log(bk) + (1-a)[(1-b)L]^{\psi} \log[(1-b)L]}{a(bK)^{\psi} + (1-a)[(1-b)L]^{\psi}}]_{\psi \to 0}$$

$$= \log(A) + a \log(bK) + (1-a) \log[(1-b)L]$$

It follows that $Y = \tilde{A} K^{a} L^{1-a}$, where $\tilde{A} = A b^{a} + (1-b)^{(1-a)}$. That is, the CES production function approaches the Cobb–Douglas form as $\psi$ tends to zero.

**c-)** Express output per capita as a function of capital per capita

**Solution:** Divide both sides of

$$Y = F(K, L) = A(a[bK]^{\psi} + (1-a)[(1-b)L]^{\psi})^{1/\psi}$$

by $L$ to get an expression for output per capita:

$$y = f(k) = A(a(bk)^{\psi} + (1-a)(1-b)^{\psi})^{1/\psi}$$

**d-)** Compute the marginal product of capital and the average product of capital

**Solution:**

$$f'(k) = Aab^{\psi}[ab^{\psi} + (1-a)(1-b)^{\psi}k^{-\psi}]^{(1-\psi)/\psi}$$

$$f(k)/k = A[ab^{\psi} + (1-a)(1-b)^{\psi}k^{-\psi}]^{1/\psi}$$

**e-)** Under what conditions of the parameters the Solow-Swan model with a CES production function displays *a steady state with zero growth*, and displays *endogenous growth*? Explain intuitively

**Solution:**
$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n)$$

Consider first the case $0 < \psi < 1$ (a high degree of substitution between L and K)

$$\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} \frac{f(k)}{k} = Aba^{1/\psi} > 0 \qquad \& \qquad \lim_{k\to 0} f'(k) = \lim_{k\to 0} \frac{sf(k)}{k} = \infty$$

If the saving rate is high enough, so that $sAab^{1/\psi} > (\delta + n)$, per capita growth rate is always positive, and the model generates endogenous, steady-state growth

Assume now $\psi < 0$, that is, a low degree of substitution between L and K.

$$\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} \frac{sf(k)}{k} = 0 \qquad \& \qquad \lim_{k\to 0} f'(k) = \lim_{k\to 0} \frac{sf(k)}{k} = Aba^{1/\psi} < \infty$$

If $sAba^{1/\psi} > \delta + n$ the model does not generate endogenous growth but there is steady state of capital. Suppose that the saving rate is low enough so that $sAba^{1/\psi} < \delta + n$. In this case, the $sf(k)/k$ curve starts at a point below $\delta + n$, and it converges to 0 as $k$ approaches zero.

## Q: The Solow-Swan Model: Conditions for the Existence of Steady State

Suppose the production function is of the form

$$F(K,L) = \frac{K}{.25L + .5K}$$

Derive the function for the steady state of capital and give conditions under which a steady state will exist. Discuss the possibility of $k$ increasing indefinitely or decreasing indefinitely (to zero).

**Solution:** Define $f(k)$ as follows

$$f(k,1) = f(\frac{K}{L},1) = \frac{K/L}{.25 \cdot 1 + .5 \cdot K/L} = \frac{k}{.25 + .5k}$$

(We can do this as $F(K,L)$ is homogeneous of degree 1 in $K$, $L$. $k = K/L$.)

For a steady state with a positive level of capital stock, it must be the case that:

$$s\frac{f(k^*)}{k^*} = n + \delta$$

Plugging in $f(k)$ we can solve for $k^*$

$$k^* = 2(\frac{s}{n+\delta} - 0.25)$$

For $k > 0$, it must be the case that

$$\frac{s}{n+\delta} > 0.25$$

The condition above implies that a positive steady state can only happen if the saving rate is high enough. This requirement come up because the production function does not satisfy all the Inada conditions

$$\lim_{K\to 0} F(K,L) = \frac{.25L}{(.25L+.5K)^2} \neq \infty$$

The other Inada condition holds

$$\lim_{K\to \infty} F(K,L) = \frac{.25L}{(.25L+.5K)^2} = 0$$

**Q: The Solow-Swan Model: Capital flows in the Solow model**

Suppose that there are two countries, Scotland and Canada. Both countries are well-described by the Solow model and have identical Cobb-Douglas production functions:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

The countries have the same level of technology ($A$) but have different capital to labor ratios, and thus different per capita income. Define the gross return on capital as $r_t + 1 - \delta$. Let $\delta = 0$ and $\alpha = 1/3$

**a-)** If Canada has 10 times Scotland's per capita income, what is the ratio of per-capita capital in Canada to that in Scotland?

**Solution:** First note that

$$\frac{y_c}{y_s} = \frac{Ak_c^{\alpha}}{Ak_s^{\alpha}}$$

and we solve for $k_c/k_s$ as a function of $y_c/y_s$ and model parameters. This give

$$\frac{k_c}{k_s} = (\frac{y_c}{y_s})^{1/\alpha} = 10^3 = 1000$$

In other words, in order to produce 10 times Scotland's output, Canada must have 1000 times Scotland's capital.

**b-)** If Canada has a 5 percent net return on capital, (and 10 times the per capita income of Scotland), what is the net return on capital in Scotland?

**Solution:** We know from the firm's profit maximization problem that

$$r = f'(k) = \alpha A k^{\alpha-1}$$

so

$$r_t = \alpha A k_s^{\alpha-1} = \alpha A(\frac{k_s}{k_c}k_c)^{\alpha-1} = (\frac{k_s}{k_c})^{\alpha-1}\alpha A k_c^{\alpha-1} = (\frac{k_c}{k_s})^{1-\alpha} r_c = 1000^{2/3}0.05 = 5$$

So Scotland has a 500% net return on capital (the trick in the solution is to find A)

**c-)** If you were a Canadian under these circumstances with some savings to invest, would you invest it in Canada or send it to Scotland?

**Solution:** Better invest in Scotland! But we don't see these kinds of returns in less-developed countries. This is because poor countries have much lower TFP (in addition to less capital) than the rich ones. The poor countries are also riskier to invest

## Q: Ramsey Model: Harrod-Domar

Production function is given as Leontief: $Y = \min(AK, BL)$. Suppose that agents are optimizing. Imagine that the dynamic system characterizing the solution is as usual under the assumption of zero growth rates of population and technology

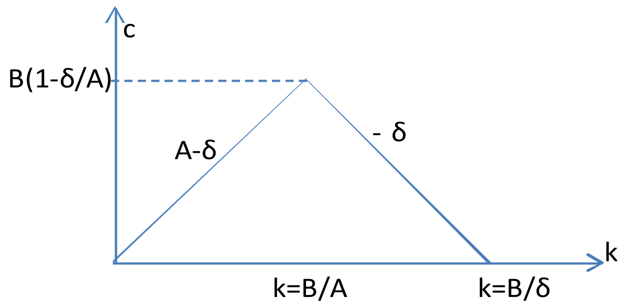
$$\dot{k} = f(k) - c - \delta k \quad \text{and} \quad \frac{\dot{c}}{c} = \frac{1}{\theta}(f'(k) - \delta - \rho)$$

**a-)** Draw the locus for $\dot{k} = 0$ on the $c - k$ space (distinguish two regions: $k \lesseqgtr B/A$)

**Solution:** $$Y = \min(AK, BL) = L \min(Ak, B) \quad \Rightarrow \quad y = \begin{matrix} Ak & \text{if } \ k < B/A \\ B & \text{if } \ k > B/A \end{matrix}$$

$$\dot{k} = \begin{matrix} Ak \\ B \end{matrix} - c - \delta k = \begin{matrix} (A-\delta)k - c \\ B - \delta k - c \end{matrix} = 0 \quad \Rightarrow \quad c = \begin{matrix} (A-\delta)k & \text{if } \ k < B/A \\ B - \delta k & \text{if } \ k > B/A \end{matrix}$$

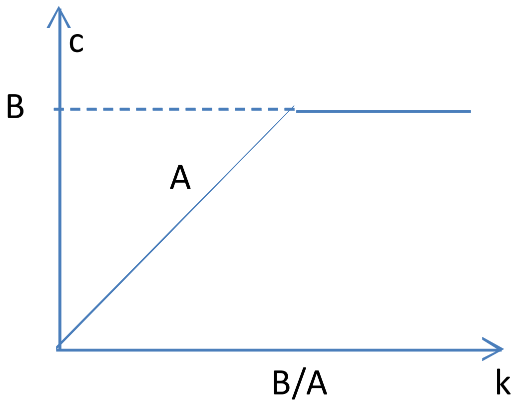
Hence, the $\dot{k} = 0$ locus can be shown as



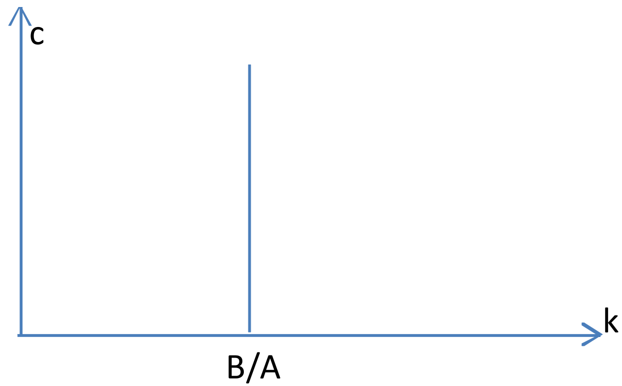
**b-)** Draw $f(k)$ as a function of $k$. Notice that the derivative $f(k)$ is not defined at $k = B/A$ (to the left of $k = B/A$ we have $A$, and to the right of $k = B/A$ we have 0), but assume that it is $\delta + \rho$. Then draw the locus for $\dot{c} = 0$ on the $c - k$ space

**Solution**: $\frac{\dot{c}}{c} = \frac{1}{\theta}(f'(k) - \delta - \rho)$; and at k=B/A we have $\frac{\dot{c}}{c} = \frac{1}{\theta}(\delta + \rho - \delta - \rho) = 0$

f(k)

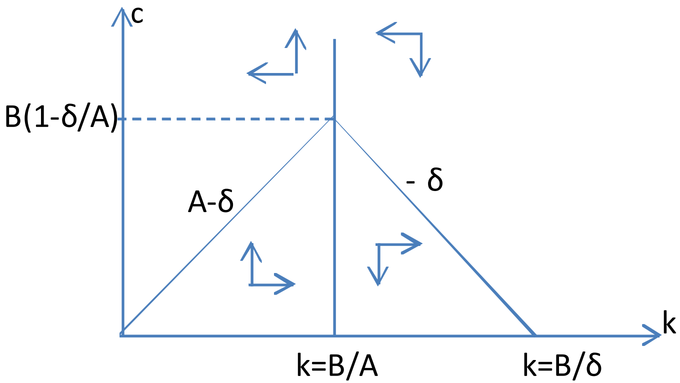



the $\dot{c} = 0$ locus



**c-)** Draw the direction of $c$ and $k$ with arrows on the four regions separated by $\dot{k} = 0$ and $\dot{c} = 0$ locuses

**Solution:** $\dot{k} = \begin{array}{ll} Ak - c - \delta k & \text{if } \; k < B/A \\ B - c - \delta k & \text{if } \; k > B/A \end{array}$ $\qquad \frac{\dot{c}}{c} = \frac{1}{\theta}(f'(k) - \delta - \rho)$




**d-)** Suppose that the steady state occurs where $\dot{k} = 0$ and $\dot{c} = 0$ locuses intersect with positive level of capital and consumption. Discuss if there is any idle factor in the economy

**Solution**: Since steady state occurs at $k = B/A$, which satisfies $AK = BL$, the production function implies that there is no idle factor in the economy

**Q: Ramsey Model**

Consider the Ramsey economy with technological growth, where the consumption and capital law of motions are given by

$$\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k} \quad \text{and} \quad \frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho - \theta x)$$

with TVC

$$\lim_{t \to \infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x] dv\} = 0$$

which implies that at the steady state

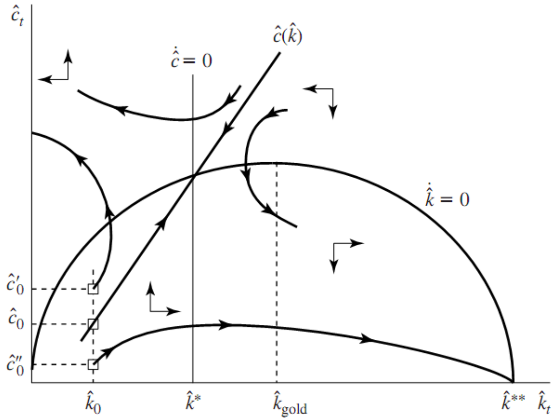
$$f(\hat{k}^*) - \delta > n + x.$$

Now assume that the model is on the balanced growth path with steady state values of $\hat{k}^*$ and $\hat{c}^*$. What are the effects of a decline in the depreciation rate, $\delta$, on the $\dot{k} = 0$ and $\dot{c} = 0$ locuses, and on the $\hat{k}^*$, $\hat{c}^*$ and $\hat{y}^*$. Explain your results and give a brief economic interpretation.

**Solution:**

Given that $\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n+\delta+x)\hat{k}, \qquad \frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho - \theta x)$

and $\lim_{t\to\infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x]dv\} = 0$




If $\delta$ declines, $\hat{c}^{\cdot} = 0$ locus shifts to the right, and $\hat{k}^{\cdot} = 0$ locus shifts upward, As a result: $\hat{k}^* \uparrow$, $\hat{c}^* \uparrow$ and $\hat{y}^* \uparrow$.

Economic Interpretation:.$\delta$.is the depreciation rate. If it declines, the economy moves towards higher steady state values.