TOBB-ETU - Econ 532 Practice Problems II (Solutions)

Q: Ramsey Model: Exponential Utility

Assume that infinite-horizon households maximize a utility function of the exponential form

$$\max U = \int_0^\infty e^{(n-\rho)t} - (1/\theta)e^{-\theta c} dt \qquad \text{where } \theta > 0$$

The behavior of firms is the same as in the Ramsey model, with zero technological progress.

a) Find the first-order conditions for a representative household with preferences given by this form of u(c).

b-) Relate θ to the concavity of the utility function and to the desire to smooth consumption over time. Compute the intertemporal elasticity of substitution. How does it relate to the level of per capita consumption, c?

c-) Combine the first-order conditions for the representative household with those of firms to describe the behavior of c and k over time. [Assume that $k(0)$ is below its steady-state value.]

d-) Derive the growth rates for the decentralized economy and for the social planner. Comment on differences between this economy and the market solution.

Solution (Decentralized Economy):

Households max. utility

$$\max U = \int_0^\infty -(1/\theta)e^{-\theta c}e^{(n-\rho)t}dt$$

subject to

$$\dot{a} = \omega + ra - c - na$$

The present value Hamiltonian

$$\mathtt{H} = -(1/\theta)e^{-\theta c}e^{(n-\rho)t} + \lambda(ra + \omega - c - na)$$

Taking FOCs

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-(\rho-n)t} = \lambda \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial a} = -\dot{\lambda} \quad \Rightarrow \quad (r-n)\lambda = -\dot{\lambda} \tag{2}$$

The Transversality Condition:

$$TVC \quad \Rightarrow \quad \lim_{t\to\infty} \lambda(t)a(t) = 0$$

Taking derivative of (1) with respect to time

$$(1) \quad \Rightarrow \; u''(c)\dot{c}e^{(n-\rho)t} - u'(c)(\rho - n)e^{-(\rho-n)t} = \dot{\lambda}$$

substitute this back into (2)

$$(r-n)u'(c)e^{-(\rho-n)t} = u''(c)\dot{c}e^{-(\rho-n)t} - u'(c)(\rho - n)e^{-(\rho-n)t}$$

which results in famous *Euler Equation* (where $u''(c) < 0$):

$$\Rightarrow \quad \frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho)$$

Using the functional form for $U(\cdot)$

$$u(c) = -(1/\theta)e^{-\theta c} \qquad \theta > 0$$

$$\Rightarrow \quad \text{Elasticity of substitution:} \; -\frac{u'(c)}{u''(c)c} = \frac{1}{\theta c}$$

Euler equation reduces to

$$\dot{c} = \frac{1}{\theta}(r - \rho) \quad \text{or} \quad \frac{\dot{c}}{c} = \frac{1}{\theta c}(r - \rho)$$

Firms maximize current profits. This brings to

$$\max_{K;L} f(k)L - (r + \delta)K - \omega L$$

FOCs find MPK and MPL:

$$\begin{aligned} \frac{\partial}{\partial K} &= 0 \implies r = f'(\hat{k}) - \delta \\ \frac{\partial}{\partial L} &= 0 \implies \omega = f(k) - f'(k)k \end{aligned}$$

Equilibrium

Closed economy $\Rightarrow \quad a = k$

The equation, $\dot{a} = \omega + ra - c - na$, can be written as

$$\dot{k} = \omega + rk - c - nk$$

plugging in for w and r

$$\dot{k} = f(k) - f'(k)k + (f'(\hat{k}) - \delta)k - c - nk = f(k) - c - (n+\delta)k \tag{3}$$

The Euler equation combined with r finds

$$\frac{\dot{c}}{c} = \frac{1}{\theta c}(f'(\hat{k}) - \delta - \rho) \tag{4}$$

Solving (3) and (4) around the s.s. (in the absence of technology)

$$\begin{aligned} \dot{c} &= 0 \quad \Rightarrow \quad f'(k) = \delta + \rho \quad \Rightarrow \quad k^*_{ss} = f_k^{-1}(\delta + \rho) \\ \dot{k} &= 0 \quad \Rightarrow \quad c^*_{ss} = f(k) - c - (n+\delta)k \end{aligned}$$

Solution (Centralized Economy):

$$\max U = \int_0^\infty e^{(n-\rho)t} - (1/\theta)e^{-\theta c} dt$$

subject to the resource constraint of the economy

$$\dot{k} = f(k) - c - nk - \delta k$$

The present value Hamiltonian is

$$\mathtt{H} = -(1/\theta)e^{-\theta c}e^{(n-\rho)t} + \lambda(f(k) - c - nk - \delta k)$$

The TVC is

$$TVC \quad \Rightarrow \quad \lim_{t\to\infty} \lambda(t)k(t) = 0$$

By using the FOC's, we can see that the command solution of this model is identical to the one obtained by the market economy. Hence

$$\begin{aligned} \dot{k} &= f(k) - c - (n+\delta)k \\ \frac{\dot{c}}{c} &= \frac{1}{\theta c}(f'(\hat{k}) - \delta - \rho) \end{aligned}$$

We can conclude that the market-competitive economy reaches the Pareto efficiency in the long-run, since it obtains the same result as the one got by the social planner

Q: Ramsey Model: Spillovers from Aggregate Capital Stock

Consider a ramsey model where the utility function of consumers has the CRRA form

$$\max U = \int_0^\infty e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt$$

Suppose the firm i's productivity depends on the economy's on the aggregate capital stock, K

$$Y_i = AK_i^\alpha L_i^{1-\alpha} K^{1-\alpha}$$

Derive the growth rates for the decentralized economy and for the social planner. Comment on differences between these two solutions. Finally discuss if the scale effect appears in the solutions

Solution (Decentralized Economy):

Households max. utility

$$\max U = \int_0^\infty e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt$$

subject to

$$\dot{a} = \omega + ra - c$$

which, as usual, finds

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$$

Firms maximize current profits. This brings to

$$\max_{K_i; L_i} AK_i^\alpha L_i^{1-\alpha} K^{1-\alpha} - (r+\delta)K_i - \omega L_i$$

FOCs find MPK and MPL:

$$\begin{aligned}
\frac{\partial}{\partial K_i} &= 0 \implies r = \alpha A k_i^{-(1-\alpha)} K^{1-\alpha} - \delta \\
\frac{\partial}{\partial L_i} &= 0 \implies \omega = (1-\alpha) A k_i^\alpha K^{1-\alpha}
\end{aligned}$$

Equilibrium

Closed economy $\Rightarrow \quad a = k \qquad and \qquad k_i = k \qquad and \quad K = kL$

The equation, $\dot{a} = \omega + ra - c$, for a closed economy can be written as

$$\dot{k} = \omega + rk - c$$

plugging in for w and r

$$\dot{k}^{Decentralized} = AkL^{1-\alpha} - c - \delta k$$

The Euler equation combined with r finds

$$\gamma_c^{Decentralized} = \frac{\dot{c}}{c} = \frac{1}{\theta}(\alpha AL^{1-\alpha} - \delta - \rho)$$

Solution (Centralized Economy):

$$\max U = \int_0^\infty e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} d$$

subject to the constraints

$$\begin{aligned} \dot{k} &= A k_i^\alpha K^{1-\alpha} - c - \delta k \\ k_i &= k \\ K &= kL \\ k(0) &= k_0 > 0 \quad \text{given} \end{aligned}$$

The present value Hamiltonian is

$$\mathtt{H} = \frac{c^{1-\theta} - 1}{1-\theta} e^{(n-\rho)t} + \lambda(A k^\alpha K^{1-\alpha} - c - \delta k)$$

Using the FOC's and the TVC,

$$\dot{k}^{Centralized} = AkL^{1-\alpha} - c - \delta k \quad \& \quad \gamma_c^{Centralized} = \frac{\dot{c}}{c} = \frac{1}{\theta}(AL^{1-\alpha} - \delta - \rho)$$

Remember that

$$\dot{k}^{Decentralized} = AkL^{1-\alpha} - c - \delta k \quad \& \quad \gamma_c^{Decentralized} = \frac{\dot{c}}{c} = \frac{1}{\theta}(\alpha AL^{1-\alpha} - \delta - \rho)$$

We can see that the market economy is not Pareto efficient since α enters the competitive growth rate of the economy. Social planner takes into account that an increase in one of the firms's capital stock increases to total production of the country through spillover effect, but firms in the market economy neglect this and tahe aggregate K as given. Moreover, the market economy and the command one have scale effects because of the presence of L within the growth rates. This is because when K rises it provides a spillover that raises the productivity of all firms

Note: The social optimum can be attained in the decentralized economy by introducing an investment-tax credit at the rate $1 - \alpha$ and financing it with a lump-sum tax. If buyers of capital pay only the fraction α of the cost, the private return on capital corresponds to the social return. Alternatively, the government could generate the same outcome by subsidizing production at the rate $(1 - \alpha)/\alpha$

Q: Ramsey Model: Optimal Growth - The Cake-Eating Problem (*You are not responsible from this example)

Consider the discrete time optimal growth problem, where

$$f(k) = k$$

This problem is commonly called a "cake-eating" problem. The consumer starts with a certain amount of capital, and "eats" it over time. The planner's problem is to maximize:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the constraints:

$$k_{t+1} \leq k_t - c_t$$
$$k_t \geq 0$$
$$k_0 > 0$$

where log is the natural (base e) logarithm function. We will use this problem to see the solution methods in dynamic programming.

a-) Write down Bellman's equation for this problem.

Solution: Bellman's equation is:

$$V(k) = \max_{c \in [0,k]} \{\log c + \beta V(k-c)\}$$

b-) First we perform policy function iteration. We start with guessing the optimal policy (control variable as a function of state variable) as

$$c_t(k) = (1-\beta)k_t$$

Write down the value of k_t (for any $t > 0$) as a function of k_0, β, and t if this policy is followed.

Solution: If this control policy is followed:

$$\begin{aligned} k_1 &= k_0 - c_0 = k_0 - (1-\beta)k_0 = \beta k_0 \\ k_2 &= \beta k_1 \\ &\ldots \\ k_t &= \beta^t k_0 \end{aligned}$$

c-) Write down the value of c_t as a function of k_0, β, and t if this policy is followed.

Solution:

$$c_t = (1-\beta)k_t = (1-\beta)\beta^t k_0$$

d-) What is the value function if the equation above describes the optimal policy? (Just to keep things simple, feel free to drop any constant term)

Solution: The value function is calculated by simply substituting the sequence $\{c_t\}$ into the utility function:

$$\begin{aligned} V(k) &= \sum_{t=0}^{\infty} \beta^t \log((1-\beta)\beta^t k) \\ &= \sum_{t=0}^{\infty} \beta^t (\log((1-\beta)\beta^t) + \log k) \\ &= \sum_{t=0}^{\infty} \beta^t (\log((1-\beta)\beta^t) + \sum_{t=0}^{\infty} \beta^t \log k \end{aligned}$$

$$V(k) = \text{constant} + \frac{1}{1-\beta} \log k$$

e-) The next step is to calculate the optimal policy under the new value function. This will give you a new policy function, which we will call $c_1(k)$. Find $c_1(k)$.

Solution: We can re-write the Bellman equation with the value function as

$$V(k) = \{\log c + \beta[\frac{1}{1-\beta}\log(k-c)]\}$$

There is a specific value of current consumption, $c_1(k)$, that maximizes

$$c_1(k) = \arg\max_{c\in[0,k]}\{\log c + \beta[\frac{1}{1-\beta}\log(k-c)]\}$$

Taking the first order condition, the new policy function can be found as

$$0 = \frac{1}{c} - \frac{\beta}{1-\beta}\frac{1}{k-c} \qquad \Rightarrow \qquad c_1(k) = (1-\beta)k$$

f-) To find the true optimal policy, you keep applying these two steps until your policy function stops changing i.e., $c_i(k) = c_{i+1}(k)$. What is the true optimal policy function?

Solution: Since $c_1(k)$ and $c_0(k)$ are identical, we do not need any further iteration. We have already found the optimal policy function as

$$c(k) = (1 - \beta)k$$

g-) Next we will try value function iteration. First we guess at the form of the value function.

Suppose that your initial guess for the value function is:

$$V_0(k) = \log k$$

Next, we calculate a new value function according to the formula:

$$V_{i+1}(k) = \max_{c \in [0,k]} \{\log c + \beta V_i(k - c)\}$$

Calculate V_1, V_2, and V_3. (Feel free to throw out any constant terms.)

Solution: First we find V_1 by solving the maximization problem:

$$V_1(k) = \max_{c \in [0,1]} \{\log c + \beta V_0(k - c)\} = \max_{c \in [0,1]} \{\log c + \beta \log(k - c)\}$$

The first order conditions are:

$$\frac{1}{c} = \frac{\beta}{k-c} \qquad \Rightarrow \qquad c = \frac{1}{1+\beta}k$$

Then we substitute back in to get V_1

$$V_1(k) = \log(\frac{1}{1+\beta}k) + \beta \log(\frac{\beta}{1+\beta}k) = \text{constant} + (1+\beta)\log k$$

Since V_1 and V_0 are not identical, we continue. Skipping through the algebra:

$$\begin{aligned} V_2(k) &= \text{constant} + (1+\beta+\beta^2)\log k \\ V_3(k) &= \text{constant} + (1+\beta+\beta^2+\beta^3)\log k \end{aligned}$$

h-) If you've done it right, you should see a pattern. Use this pattern to discern V_i for an arbitrary integer i.

Solution: The pattern should be clear at this point:

$$V_i(k) = \text{constant} + \sum_{j=0}^{i} \beta_j \log k$$

ı-) What is the limit of this as $i \to \infty$?

Solution: The limit is:

$$V(k) = \text{constant} + \frac{1}{1-\beta}\log k$$

i-) What is the optimal policy function when this is the value function?

Solution:

$$V_1(k) = \max_{c\in[0,1]}\{\log c + \beta V_0(k-c)\} = \max_{c\in[0,1]}\{\log c + \frac{\beta}{1-\beta}\log(k-c)\}$$

The first order conditions are

$$\frac{1}{c} = \frac{\beta}{1-\beta}\frac{1}{k-c}$$

Not too surprisingly, the optimal policy under this value function is:

$$c(k) = (1-\beta)k$$

j-) Try to solve the same problem the way we used in the lecture notes

Solution: The Bellman equation is

$$V_t(k_t) = \max_c \{\log c_t + \beta V_{t+1}(k_{t+1})\}$$

subject to

$$k_{t+1} = k_t - c_t$$

- Some additional Information:
  - All CRRA, CARA, and quadratic utility functions are the class of HRRA (Hyperbolic absolute risk aversion) utility function
  - Merton shows that the value function is of the same functional form as the utility function for the HRRA utility functions if labor income is fully diversified, or where there is no labor income at all and all income is derived from tradable wealth.

Hence, assuming the same functional form for the value function with the utility function

$$V_t(k_t) = A \log k_t + B$$

the Bellman equation reduces to

$$A \log k_t + B = \log \max_{c_t} \{ \log c_t + \beta (A \log k_{t+1} + B) \}$$

FOC wrt c_t

$$0 = \frac{1}{c_t} + A\beta \frac{-1}{k_t - c_t} \tag{1}$$

FOC wrt k_t

$$\frac{1}{k_t} = \beta \frac{1}{k_t - c_t} \tag{2}$$

(1) and (2) imply that $c_t = k_t / A$. Using this equality in (2)

$$\frac{1}{k_t} = \beta \frac{1}{k_t - A k_t} \quad \Rightarrow \quad A = 1/(1-\beta) \quad \Rightarrow \quad c(k) = (1-\beta)k$$

Q: Ramsey Model: Optimal growth with Logarithmic utility

Consider an optimal growth model with 100% depreciation every period. The social planner maximizes:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the constraints:

$$k_{t+1} = Ak_t^{\alpha} - c_t$$
$$k_t \geq 0$$
$$k_0 > 0$$

a-) Write down the Bellman equation for this planner's problem.

Solution: The Bellman equation can, for instance, be written as:

$$V(k_t) = \max\{\log(Ak_t^{\alpha} - k_{t+1}) + \beta V(k_{t+1})\}$$

b-) Write down the necessary conditions for a solution to this planner's problem

Solution: The way that we've set up the Bellman equation, the necessary conditions are the FOCs w.r.t. k_t and k_{t+1}

$$\begin{aligned}\frac{1}{Ak_t^\alpha - k_{t+1}} &= \beta V'(k_{t+1}) \\ V'(k_t) &= \alpha A k_t^{\alpha-1} \frac{1}{Ak_t^\alpha - k_{t+1}}\end{aligned}$$

c-) Derive the Euler equation.

Solution: The Euler equation is:

$$\frac{\beta \alpha A k_{t+1}^{\alpha-1}}{c_{t+1}} = \frac{1}{c_t}$$

d-) Find the steady state values of k_t, c_t, and y_t.

Solution: The steady-state capital, output, and consumption are:

$$
\begin{aligned}
k^* &= (\alpha\beta A)^{1/(1-\alpha)} \\
y^* &= A(k^*)^\alpha = A(\alpha\beta A)^{\alpha/(1-\alpha)}
\end{aligned}
$$

and using the dynamic equation for capital that is $k_{t+1} = Ak_t^\alpha - c_t$

$$
c^* = y^* - k^* = A(\alpha\beta A)^{\alpha/(1-\alpha)} - (\alpha\beta A)^{1/(1-\alpha)}
$$

e-) Find the steady state savings rate.

Solution: The steady-state savings rate is:

$$
\begin{aligned}
s^* &= \frac{y^* - c^*}{y^*} \\
&= \frac{(\alpha\beta A)^{1/(1-\alpha)}}{A(\alpha\beta A)^{\alpha/(1-\alpha)}} \\
&= \alpha\beta
\end{aligned}
$$

f-) Suppose that $\alpha = 1$, so that we have an "AK" model. Ignore for the time being the possibility that there is no solution and assume that the Euler equation applies. What is the growth rate of consumption?

Solution: When $\alpha = 1$, the Euler equation becomes:

$$\frac{\beta A}{c_{t+1}} = \frac{1}{c_t}$$

We can rearrange so that:

$$\frac{c_{t+1}}{c_t} = \beta A$$

In general, the growth rate of a variable x_t is $x_{t+1}/x_t - 1$ (by definition) or $\log(x_{t+1}/x_t)$ (a convenient approximation), so the growth rate of consumption is:

$$\beta A - 1$$

or

$$\log(\beta A)$$

Q: Exercise on Income Taxation in the Ramsey Model

Consider a ramsey model where the utility function of consumers has the CRRA form

$$\max U = \int_0^\infty e^{-(\rho-n)t} \frac{c^{1-\theta}-1}{1-\theta} dt$$

Government taxes family's incomes and the revenue so collected will be distributed to households in the form of lump-sum transfers.

$$\dot{a} = (1-\tau)(\omega + ra) - c - na + z$$

Derive the growth rates for the decentralized economy and for the social planner. Comment on differences between these two solutions.

Solution (Decentralized Economy):

Households max. utility

$$\max U = \int_0^\infty e^{-\rho t} \frac{c^{1-\theta} - 1}{1 - \theta} dt$$

subject to

$$\dot{a} = (1 - \tau)(\omega + ra) - c - na + z$$

where z is considered as constant, so that agents take its value as a given and represents transfers made by the social planner to households. Solving the hamiltonian, as usual, finds

$$\frac{\dot{c}}{c} = \frac{1}{\theta}((1 - \tau)r - \rho)$$

Since the firms' production function is not given, we just assume that

$$y = f(k)$$

Firms maximize current profits

$$\max_{K;L} f(k)L - (r+\delta)K - \omega L$$

FOCs find that

$$\begin{aligned} \frac{\partial}{\partial K_i} &= 0 \implies r = f'(k) - \delta \\ \frac{\partial}{\partial L_i} &= 0 \implies \omega = f(k) - f'(k)k \end{aligned}$$

Equilibrium

$$\begin{aligned} a &= k \\ z &= \tau(rk+\omega) \end{aligned}$$

Using $a = k$ in the households' budget constraint together with w and r

$$\dot{k} = f(k) - c - nk - \delta k$$

The Euler equation combined with r finds

$$\frac{\dot{c}}{c} = \frac{1}{\theta}((1-\tau)f'(k) - \delta - \rho)$$

As a result we can find steady states as

$$\begin{aligned} k_{ss}^{**} &= f_k^{-1}(\frac{\delta+\rho}{1-\tau}) \\ c_{ss}^{**} &= f(k^{**}) - c - (n+\delta)k^{**} \end{aligned}$$

We can see that with income taxation in the market solution, the steady state value of capital per-capita depends negatively on the tax rate τ. This means that the $\dot{c} = 0$ locus shifts towards left compare to its position in the original Ramsey model, which results in a reduction of the equilibrium steady-state value of the capital-labour ratio: a distortion. *Since there can be no taxes in the social planner solution, the solution of the centralized economy is the same as with the basic Ramsey model.*

Q: Ramsey Model: Congestion of Public Services (G/Y)

Consider a ramsey model where the utility function of consumers has the CRRA form

$$\max U = \int_0^\infty e^{-(\rho-n)t}\frac{c^{1-\theta}-1}{1-\theta}dt$$

Suppose that output for firm i is given by

$$Y_i = AK_i(g/y)^\alpha$$

and for single firm

$$y_i = Ak_i(g/y)^\alpha$$

that is governmental activities (highways, water systems, police and fire services, and courts) serve as an input to private production. G is financed by a proportional tax on households' earnings

$$\dot{a} = (1-\tau)(\omega + ra) - c - na$$

This final equation also states that $g/y = \tau$ and τ is fixed. Derive the growth rates for the decentralized economy and for the social planner. (Assume that the social planner takes g/y given as well.) Comment on differences between these two solutions.

Solution (Decentralized Economy):

Households max. utility

$$\max U = \int_0^\infty e^{-(\rho-n)t} \frac{c^{1-\theta}-1}{1-\theta} dt$$

subject to

$$\dot{a} = (1-\tau)(\omega + ra) - c - na$$

Solving the hamiltonian, as usual, finds

$$\frac{\dot{c}}{c} = \frac{1}{\theta}((1-\tau)r - \rho)$$

Firms maximize current profits

$$\max_{K_i;L_i} AK_i(g/y)^\alpha - (r+\delta)K_i - \omega L_i$$

FOCs find that

$$\frac{\partial}{\partial K_i} = 0 \implies r = A(g/y)^\alpha - \delta$$

$$\frac{\partial}{\partial L_i} = 0 \implies \omega = 0$$

Equilibrium

$$\begin{aligned} a &= k \quad , \quad k_i = k \quad , \quad y_i = y \\ \dot{k} &= (1-\tau)(\omega + rk) - c - nk \\ g/y &= \tau \end{aligned}$$

Inserting g/y in the interest rate equation

$$r = A(g/y)^\alpha - \delta = A\tau^\alpha - \delta$$

Using $a = k$ in the households' budget constraint together with w and r

$$\dot{k}^{Decentralized} = (1-\tau)A\tau^\alpha k - c - nk - \delta k$$

The Euler equation combined with r finds

$$\gamma_c^{Decentralized} = \frac{1}{\theta}[(1-\tau)A\tau^\alpha - \delta - \rho]$$

since $\tau\delta \to 0$

Solution (Centralized Economy):

Social planner seeks to maximize the utility function

$$\max U = \int_0^\infty e^{(n-\rho)t}\frac{c^{1-\theta}-1}{1-\theta}dt$$

subject to the constraints

$$\begin{aligned} \dot{k} &= Ak_i(g/y)^\alpha - c - g - nk - \delta k \\ k(0) &= k_0 > 0 \quad \text{given} \end{aligned}$$

The present value Hamiltonian is

$$\mathtt{H} = \frac{c^{1-\theta}-1}{1-\theta}e^{(n-\rho)t} + \lambda(Ak_i\tau^\alpha - c - g - nk - \delta k)$$

Using the FOC's and the TVC

$$\begin{aligned} \dot{k}^{Centralized} &= A\tau^{\alpha}k - c - g - nk - \delta k \\ \gamma_c^{Decentralized} &= \frac{1}{\theta}[A\tau^{\alpha} - \delta - \rho] \end{aligned}$$

Remember that

$$\begin{aligned} \dot{k}^{Decentralized} &= (1-\tau)A\tau^{\alpha}k - c - nk - \delta k \\ \gamma_c^{Decentralized} &= \frac{1}{\theta}[(1-\tau)A\tau^{\alpha} - \delta - \rho] \end{aligned}$$

We know that $y = A\tau^{\alpha}k$; hence, we can conclude that $g = \tau A\tau^{\alpha}k$ so that the growth rates of capital are equal to each other in both scenarios. On the other hand, the growth rates of consumption differ across solutions. The reason is that in the market economy consumers face with lower return to their savings due to the distortionary taxation.

Q: OLG Model: Dynamic inefficiency

Remember the OLG model we solved in the class with a Cobb-Douglas production function and log-utility. Suppose population growth rate, n, is 0 and depreciation rate, δ, is 1.

a-) Solve for the worker's savings rate.

Solution: First we note that the first order conditions imply:

$$\frac{c_{1,t}}{c_{2,t+1}} = \frac{1+\rho}{1+r_{t+1}}$$

Since $c_{2,t+1} = (1 + r_{t+1})s_t$, we have:

$$c_{1,t} = s_t(1+\rho)$$

Since $s_t + c_{1,t} = w_t$, we have

$$s_t = \frac{1}{2+\rho} w_t$$

The saving rate is

$$s = \frac{1}{2+\rho}$$

b-) Solve for the golden rule level of the capital stock.

Solution: Calculating the resource constraint of the economy, that is

$$k_{t+1}(1+n) - k_t = f(k_t) - c_t - \delta k_t$$

at the steady state with $n = 0$ and $\delta = 1$ results in

$$c^* = A(k^*)^\alpha - k^*$$

The golden rule is the value of k^* that maximizes c^*. The first order conditions give

$$0 = \alpha A k^{\alpha - 1} - 1$$

which results in

$$k_{GR} = (\alpha A)^{1/(1-\alpha)}$$

c-) Solve for the steady-state savings rate needed to maintain the golden rule level of the capital stock.

Solution: The savings rate of young worker at the steady state is (notice that $s_t = (1+n)k_{t+1}$ and n is 0 in this question):

$$s_{GR} = k_{GR}$$

$$s.rate_{GR} = \frac{k_{GR}}{w_{GR}} = \frac{k_{GR}}{f(k_{GR}) - f'(k_{GR})k_{GR}} = \frac{k_{GR}}{(1-\alpha)Ak_{GR}^{\alpha}} = \frac{(\alpha A)}{(1-\alpha)Ak_{GR}^{\alpha}} = \frac{\alpha}{1-\alpha}$$

Notice that the golden rule savings rate is $\alpha/(1-\alpha)$ fraction of w, which is the total income of youth. However, since $w = (1-\alpha)y$, the golden rule savings rate is α fraction of y, just like the Solow model

d-) Under what conditions on the model parameters will the savings rate exceed the golden rule level?

Solution: The equilibrium savings rate will exceed the golden rule level if

$$\frac{1}{2+\rho} > \frac{\alpha}{1-\alpha}$$

When this condition is fulfilled, the economy is in the dynamically inefficient region. This situation arises from the fact consumers want to smooth their consumption between two time priods (ρ), which would induce them to save more than efficient level of saving rate that is implied by the capital share in the production (α). So their consumption smoothing incentive prevents them from attaining the maximum of the total consumption of both periods.

e-) Prove that these conditions imply that the equilibrium is inefficient by finding a Pareto superior allocation.

Solution: In order to find a Pareto dominant allocation, we must find an allocation that leaves everyone at least as well off and leaves someone better off. Let s_E be the equilibrium savings rate and let s_{GR} be the golden rule savings rate, and we know that $k_E^* > k_{GR}$ when $s_E > s_{GR}$

Suppose $k_E^* \geqslant k(0) > k_{GR}$. Then reducing s_E to s_{GR} increase consumption for all generations. Hence, the resulting allocation would be a Pareto superior one

Suppose $k_E^* \geqslant k_{GR} > k(0)$. Then save s_E of the worker's income up until $k(0)$ reaches k_{GR}. Once k_{GR} is reached, save s_{GR} from then on. In the first period that s_{GR} is the savings rate, consumption is higher than in the equilibrium allocation, and it is higher in every subsequent period. Before this period, consumption is the same as in the equilibrium allocation. This allocation Pareto dominates the equilibrium allocation, therefore the equilibrium allocation is not Pareto efficient. (Notice that we needed to wait until the golden rule capital stock was reached to impose the golden rule savings rate. Otherwise, consumption might not be as high for some agents in the first few periods, and our allocation would fail to be Pareto dominant.)

f-) Put this result in words. I suggest something of the form "Dynamic inefficiency is more likely if capital's share is (low?/high?) and if young people care (more?/less?) about their future consumption."

Solution: Dynamic inefficiency is more likely if capital's share is low and if young people care more about their future consumption

g-) Will this model exhibit multiple equilibria? How do you know this?

Solution: The savings rate is constant. As a result, we can write k_{t+1} as a function of k_t. Therefore the equilibrium is unique

h-) Will this model exhibit history-dependence? How do you know this?

Solution: The model doesn't exhibit history dependence because we found a unique steady state (not counting the zero-capital steady state)

Q: OLG Model: Social Security using Taxes

Consider an OLG model where rate of technological progress is zero, production is Cobb-Douglas, population grows at a rate n, and utility is logarithmic in a way that

$$\sum_{t=0}^{\infty}(\frac{1}{1+\rho})^t \log c_t.$$

Suppose the government taxes each young individual an amount T and uses the profits to pay benefits to old individuals; thus each old person receives $(1+n)T$

a-) Write down the budget constraint faced by an individual born at time t.

Solution: We can write down the budget constraint as follows

$$c_t + s_t = w_t - T$$
$$c_{t+1} = (1 + r_{t+1})s_t + (1 + n)T$$

Note that, for each individual, the return on their tax, T, is $(1 + n)T$, reflecting the fact that there are more individuals in the younger generation whose taxes are being transferred to the older generation. Now, since $s_t = w_t - T - c_t$, we can substitute to solve for the intertemporal budget constraint and find

$$c_t + \frac{c_{t+1}}{1 + r_{t+1}} = w_t - T(\frac{r_{t+1} - n}{1 + r_{t+1}})$$

b-) Use the budget constraint to determine the saving function of this individual.

Solution: We know from the lecture notes that (the FOCs do not change with a lump-sum tax)

$$\frac{c_{1,t}}{c_{2,t+1}} = (\frac{1+\rho}{1+r_{t+1}})^{1/\theta}$$

Using the condition that $\theta = 1$ with the log-utility, the budget constraint reduces to

$$\begin{aligned} c_t + \frac{c_{t+1}}{1+r_{t+1}} &= w_t - T(\frac{r_{t+1}-n}{1+r_{t+1}}) \\ c_t + \frac{c_t}{1+\rho} &= w_t - T(\frac{r_{t+1}-n}{1+r_{t+1}}) \\ c_t &= (\frac{1+\rho}{2+\rho})(w_t - T(\frac{r_{t+1}-n}{1+r_{t+1}})) \end{aligned}$$

We can then substitute for savings

$$\begin{aligned} s_t &= w_t - T - c_t \\ &= w_t - T - (\frac{1+\rho}{2+\rho})(w_t - T(\frac{r_{t+1}-n}{1+r_{t+1}})) \\ &= \frac{w_t}{2+\rho} - Z_t T \qquad \text{where } \; Z_t = 1 - (\frac{1+\rho}{2+\rho})(\frac{r_{t+1}-n}{1+r_{t+1}}) \end{aligned}$$

c-) How, if at all, does the introduction of the pay as you go social security affect the relation between k_t and k_{t+1} as we found in the class [Hint: Use the individual savings function obtained in part (b) and proceed as usual to determine the relationship between k_t and k_{t+1}.]

Solution: As usual in this model,

$$\begin{aligned} K_{t+1} &= s_t L_t \\ k_{t+1} &= \frac{s_t}{1+n} \\ &= \frac{1}{1+n}(\frac{w_t}{2+\rho} - Z_t T) \end{aligned}$$

Now, with Cobb-Douglas production, the wage is $w_t = k_t^\alpha - \alpha k_t^\alpha$, so we have

$$k_{t+1} = \frac{1}{1+n}(\frac{(1-\alpha)k_t^\alpha}{2+\rho} - Z_t T)$$

d-) How, if at all, does the introduction of this social security system affect the balanced growth path value of k?

Solution: Without any taxation or social security, the relation between k_t and k_{t+1} would be just as it is above but without the $Z_t T$ term. If Z_t is positive, this means we have shifted down the $k_{t+1} = m(k_t)$ curve relative to the no-tax case. In turn, this reduces the steady state value of k. It follows that we need to do is find the sign of Z_t.

$$\begin{aligned} Z_t &= 1 - (\frac{1+\rho}{2+\rho})(\frac{r_{t+1}-n}{1+r_{t+1}}) \\ &= \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} \\ &= \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0 \end{aligned}$$

It follows that k^* is reduced relative to the case with no social security

e-) If the economy is initially on a balanced growth path (BGP) that is dynamically efficient, how does a marginal increase in T affect the welfare of current and future generations? What happens if the initial BGP is dynamically inefficient?

Solution: A dynamically efficient economy is one in which saving lies below the golden rule rate of savings. It is "dynamically efficient" in a Pareto sense; that is, it is not possible to make someone better-off without making another worse-off.

Hence, a marginal increase in T reduces savings and consumption of the young, and there is no way for future generations to compensate the present generations for this fall in consumption, and therefore the economy is still Pareto efficient. Also since savings were below the golden rule already, now future generations have even lower consumption. The current old generation, however, obviously benefits, since it gets the benefit of the increased contributions to social security.

If the economy were originally dynamically inefficient, the marginal increase in T would increase the welfare of both the current and future generations. As in this point, $r_{t+1} = f'(k_{t+1}) < n + \delta$, transferring the resources is more efficient than the using them for saving.