TOBB-ETU, Economics Department
Macroeconomics II (ECON 532)
Spring 2015-16- Practice Problems III (Solutions)

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Q: Consumption Theory (CARA utility)

Consider an individual living for two periods, with preferences

$$U(c_1, c_2) = u(c_1) + \frac{1}{1+\rho} u(c_2), \quad \text{where } u(c) = -\frac{1}{\gamma} \exp(-\gamma c) \tag{1}$$

The individual receives a deterministic income $\{y_1, y_2\}$ in the two periods. Saving/borrowing can be made at the interest rate r.

a) Write down the dynamic maximization problem of the individual as a Lagrangian and derive the Euler equation.

Solution: The consumer max (1) subject to

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{2}$$

which can be written as

$$\max_{\{c_1,c_2\}} \quad -\frac{1}{\gamma}\exp(-\gamma c_1) - \frac{1}{(1+\rho)\gamma}\exp(-\gamma c_2) + \lambda(y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r})$$

The first order conditions (necessary and sufficient) are:

$$e^{-\gamma c_1} = \lambda \qquad \frac{1}{1+\rho}e^{-\gamma c_2} = \frac{\lambda}{1+r}$$

The Euler equation is

$$\frac{1}{1+\rho}e^{-\gamma c_2} = \frac{e^{-\gamma c_1}}{1+r}$$

b) Using the approximation $ln(1+x) \simeq x$, solve for the optimal consumption choices $\{c_1^*, c_2^*\}$ as a function of $(\gamma, r, \rho, y_1, y_2)$. Under which condition is c_2^* larger than c_1^*?

Solution: Taking natural logs, we obtain

$$\gamma(c_2 - c_1) = \ln(1+r) - \ln(1+\rho)$$

Using, $ln(1+x) \simeq x$

$$\gamma(c_2 - c_1) = r - \rho \quad \text{or} \quad c_1 = c_2 + \frac{\rho - r}{\gamma}$$

When $\rho = r$ we have a constant consumption profile or perfect consumption smoothing. When $\rho < r$, then $c_1 > c_2$. To solve for the optimal level of consumption we use the IBC (Intertemporal Budget Constraint)

$$c_2 + \frac{\rho - r}{\gamma} + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

which reduces to

$$c_2^* = \frac{1+r}{2+r}[y_1 + \frac{y_2}{1+r} - \frac{\rho - r}{\gamma}]$$

and c_1^* can be found as

$$c_1^* = \frac{1+r}{2+r}[y_1 + \frac{y_2}{1+r} - \frac{\rho - r}{\gamma}] + \frac{\rho - r}{\gamma}$$

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Q: Linear Utility

Consider the intertemporal consumption-savings problem of an infinitely-lived consumer with assets A_0 at the beginning of period zero. This consumer receives a fixed return R on her assets that is paid out at the end of the period. He also has a stochastic labor income y_t that is realized before the consumption choice in every period, subject to the usual No-Ponzi GC and a non-negativity constraint on consumption, $c_t \geqslant 0$ for all t. Suppose the consumer has linear preferences over consumption, $u(c) = \gamma c$. What is the optimal path for consumption if $\beta R < 1$, $\beta R > 1$ and $\beta R = 1$? Explain.

Solution: With linear utility the intertemporal elasticity of substitution is zero so there is no motive for consumption smoothing. Therefore, with $\beta R < 1$, the optimal consumption path is to consume as much as you can in period zero, and pay back your debt during the rest of your live. With $\beta R > 1$, the optimal consumption path is to consume zero in every period and to save all your income to consume later. With $\beta R = 1$, the consumer is indifferent between all feasible consumption paths.

Q: Quadratic Utility and Fixed Income

Let's assume that $\rho = r$, and consumers preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions together with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and assuming that $y_t = \bar{y}$,

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, \bar{y})$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t y_{t+i} + (1+r) A_t$$

and interpret your finding

Solution

Using $E_t c_{t+1} = c_t$ and $y_t = \bar{y}$ together with the PVBC defined above, finds

$$\sum_{i=0}^{\infty} (\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty} (\frac{1}{1+r})^i \bar{y} + (1+r) A_t$$

since $r > 0$, it must be true that $\dfrac{1}{1+r} < 1$. Then the above equality can be written as

$$\frac{1}{1 - \dfrac{1}{1+r}} c_t = \frac{1}{1 - \dfrac{1}{1+r}} \bar{y} + (1+r) A_t$$

which can be simplified to

$$c_t = \bar{y} + r A_t$$

Interpretation: Since households earn fixed income each period and they have quadratic preferences for consumption, they do not save, and consume the same amount in each period, which is yearly labor income plus the yearly rate of return (annuity value) on wealth

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$, and interpret your findings

Solution

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (\bar{y} + rA_t) = 0$$

The transitory income is the saving, i.e. the difference between disposable income and the permanent income. If there is no saving, it must be equal to zero

Finding the relation between A_t and A_{t+1}

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (\bar{y} + rA_t) = A_t$$

Because households consume annuity value of their financial wealth, rA_t, the level of financial wealth in real terms does not change over periods, thus $A_{t+1} = A_t$

Q: Quadratic Utility and Expected Permanent Change in Income

Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we use these two assumptions with the Euler Equation:

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and also assuming that

$$E_t(y_{t+i}) = \begin{cases} \bar{y} & i = 0 \\ 2\bar{y} & i = 1, 2, ...\infty \end{cases}$$

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, \bar{y})$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding

Solution: Using $E_t c_{t+1} = c_t$ and the defined time path of y_{t+i} together with the PVBC

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \bar{y} + \sum_{i=1}^{\infty}(\frac{1}{1+r})^i 2\bar{y} + (1+r)A_t$$

which can be written as

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i 2\bar{y} - \bar{y} + (1+r)A_t$$

since $r > 0$, it must be true that $\dfrac{1}{1+r} < 1$. As a result, the above equality can be written as

$$\frac{1}{1-\frac{1}{1+r}}c_t = \frac{1}{1-\frac{1}{1+r}}2\bar{y} - \bar{y} + (1+r)A_t$$

which can be simplified to

$$c_t = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t$$

Interpretation: Since households have quadratic preferences for consumption, they prefer to consume the same amount each period as long as there is no unexpected change in future income. Hence, even though the change in income is expected to take place in the future, the level of consumption is fixed over periods, and it is slightly less than $2\bar{y}$ (It is less because s/he gains $2\bar{y}$ on every period other than today. It is slightly less because assuming the real interest rate is small, $r/(1+r)$ is a small number)

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} under this case by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$ and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t) = -\bar{y} + \frac{r}{1+r}\bar{y} = -\frac{1}{1+r}\bar{y}$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t) = A_t - \frac{1}{1+r}\bar{y}$$

Interpretation: Since at time t income gain is less than the future incomes, households, which have quadratic preferences and like to smooth their consumption pattern, either borrow today or use some of their financial wealth to smooth their consumption. As a result we have found dissaving and that consumers financial wealth has declined over two time periods.

Q: Quadratic Utility Function and Unexpected Permanent Change in Income

Let's assume that everything is the same with previous question, except that the consumer does not know at time t whether her/his income will change to $2\bar{y}$. So

$$E_t(y_{t+i}) = \bar{y} \quad for \;\; i = 1, 2, ...\infty$$

but the reality is

$$y_{t+i} = \begin{cases} \bar{y} & i = 0 \\ 2\bar{y} & i = 1, 2, ...\infty \end{cases}$$

and finally, once his income is increased to $2\bar{y}$ at time $t+1$, he changes his expectations as

$$E_{t+1}(y_{t+i}) = 2\bar{y} \qquad i = 2, ...\infty$$

a-) Find the difference between consumptions at time $t+1$ and at time t in terms of wealth and income of the consumers, $c_{t+1}(A_{t+1}, \bar{y}) - c_t(A_t, \bar{y})$, by using the following Present Value Budget Constraint (Hint: you need to solve this constraint both at time t and $t+1$)

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding

Solution: Just like the 1st problem of problem session 2, if consumer thinks that s/he is going to earn $\bar{y}$, s/he consumes

$$c_t = \bar{y} + rA_t$$

when s/he updates her/his expectations, he starts to consume

$$c_{t+i} = 2\bar{y} + rA_t \qquad i = 2, ...\infty$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint at time t, $A_{t+1} = (1+r)A_t + y_t - c_t)$, and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (\bar{y} + rA_t) = 0$$

To find the relation between A_t and A_{t+1} we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (\bar{y} + rA_t) = A_t$$

Since saving equals to zero, the wealths at time t and $t+1$ are the same

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Extra: Permanent Income Hypothesis: Quadratic Utility Function and Expected Temporary Change in Income

Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and also assuming that

$$E_t(y_{t+i}) = \begin{cases} 2\bar{y} & i = 0, 1, 3, 4 ... \infty \\ \bar{y} & i = 2 \end{cases}$$

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, y_t)$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your result

Solution: Use $E_t c_{t+1} = c_t$ and the condition for y_{t+i} defined above together with the PVBC

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i 2\bar{y} - \frac{\bar{y}}{(1+r)^2} + (1+r)A_t$$

since $r > 0$ and so $\dfrac{1}{1+r} < 1$. As a result, the above equality can be written as

$$\frac{1}{1-\dfrac{1}{1+r}} c_t = \frac{1}{1-\dfrac{1}{1+r}} 2\bar{y} - \frac{\bar{y}}{(1+r)^2} + (1+r)A_t$$

which can be simplified as

$$c_t = 2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t)$), and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (2\bar{y} + rA_t) - (2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t) = \frac{r\bar{y}}{(1+r)^3}$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + 2\bar{y} - (2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t) = A_t + \frac{r\bar{y}}{(1+r)^3}$$

Extra: Permanent Income Hypothesis: Quadratic Utility Function and Unexpected Temporary Change in Income

Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Now we use an individual who assumes that

$$E_t(y_{t+i}) = 2\bar{y} \quad for \quad i = 0, 1, 2, ...\infty$$

but the reality is

$$y_{t+i} = \begin{cases} 2\bar{y} & i = 0, 2, 3...\infty \\ \bar{y} & i = 1 \end{cases}$$

and finally even though his income decreases to $\bar{y}$ at time $t+1$, he will realize that this is a temporary change and as a result he does not change his expectations about his future income

$$E_{t+1}(y_{t+2+i}) = 2\bar{y} \quad i = 0, ...\infty$$

a-) Find the difference between consumptions at time $t+1$ and at time t in terms of wealth and income of the consumers, $c_{t+1}(A_{t+1}, y_{t+1}) - c_t(A_t, y_t)$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding (Hint: you need to solve this constraint both at time t and $t+1$)

Solution: If consumer thinks that s/he is going to earn $\bar{y}$, s/he consumes

$$c_t = 2\bar{y} + rA_t$$

when s/he updates her/his expectations, he starts to consume

$$\sum_{i=0}^{\infty} (\frac{1}{1+r})^i c_{t+1+i} = \sum_{i=0}^{\infty} (\frac{1}{1+r})^i 2\bar{y} - \bar{y} + (1+r)A^*_{t+1}$$

$$c_{t+1} = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA^*_{t+1}$$

$$c_{t+i} = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA_{t+i} \qquad i = 2, ...\infty$$

$$c_{t+1}(A_{t+1}, y_{t+1}) - c_t(A_t, y_t) = -\frac{r}{1+r}\bar{y}$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$, and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (\bar{y} + rA_t) = 0$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (\bar{y} + rA_t) = A_t$$

Q: Dynamic Programming: Quadratic Utility and no Labor Income

Remember the Bellman Equation

$$V_t[(1+r)A_t + y_t] = \max_{c_t}\{u(c_t) + \frac{1}{1+\rho}E_t V_{t+1}[(1+r)A_{t+1} + y_{t+1}]\}$$

which is subject to

$$A_{t+1} = (1+r)A_t + y_t - c_t$$

Now suppose that there is no labor income and all income is derived from tradable wealth. Further suppose that $r = \rho$ and the utility of the consumer is of the functional form: $u(c) = c_t - b/2 \cdot c_t^2$. Use the Bellman equation to solve consumption, c, in terms of A

- Some additional Information:
  - All CRRA, CARA, and quadratic utility functions are the class of HRRA (Hyperbolic absolute risk aversion) utility function

- Merton shows that the value function is of the same functional form as the utility function for the HRRA utility functions if labor income is fully diversified, or where there is no labor income at all and all income is derived from tradable wealth

As a result, we use a following functional form for $V(\cdot)$

$$V_t(A_t) = aA_t^2 + dA_t + e$$

Then the Bellman equation

$$V_t(A_t) = \max_{c_t}\{u(c_t) + \frac{1}{1+\rho} E_t V_{t+1}(A_{t+1})$$

can be modified as (there is no uncertainty, so no need for E_t)

$$aA_t^2 + dA_t + e = \max_{c_t}\{c_t - b/2 \cdot c_t^2 + \frac{1}{1+\rho}[aA_{t+1}^2 + dA_{t+1} + e]$$

that is subject to

$$A_{t+1} = (1+r)A_t - c_t$$

FOC w.r.t. A_t gives

$$2aA_t + d = \frac{1+r}{1+\rho}(2aA_{t+1} + d)$$

Assuming $r = \rho$ and using the Budget Constraint

$$2aA_t + d = (2a[(1+r)A_t - c_t] + d)$$

which can be simplified to

$$A_t = [(1+r)A_t - c_t] = A_t - rA_t - c_t$$

then

$$c_t = rA_t$$

With quadratic utility and no labor income, households consume annuity value of their financial wealth

Note: Taking derivative only with respect to A_t but not c_t has turned out to be sufficient under this case

Q: Dynamic Programming with Log Utility

Remember the Bellman Equation

$$V_t[(1+r)A_t + y_t] = \max_{c_t}\{u(c_t) + \frac{1}{1+\rho}E_t V_{t+1}[(1+r)A_{t+1} + y_{t+1}]\}$$

which is subject to

$$A_{t+1} = (1+r)A_t + y_t - c_t$$

Suppose that there is no labor income and all income is derived from tradable wealth. And suppose that the utility of the consumer is of the functional form: $u(c) = \log c$. Use the Bellman equation to solve consumption, c, in terms of A

Solution: Remember that the logarithmic utility function has CRRA form

$$u'(c) = -\frac{1}{c} \quad u''(c) = -\frac{1}{c^2} \qquad \Rightarrow \qquad -\frac{u''(c)c}{u'(c)} = 1 \quad -\frac{u''(c)}{u'(c)} = \frac{1}{c}$$

Therefore we can use the Merton result that the value function, V ( ), is of the same functional form as the utility function. So we use

$$V_t(A_t) = a \ln A_t + b$$

Then the Bellman equation

$$V_t(A_t) = \max_{c_t}\{\log(c_t) + \frac{1}{1+\rho}E_t V_{t+1}(A_{t+1})\}$$

which can be modified as (there is no uncertainty so no need for E_t)

$$a \ln A_t + b = \max_{c_t}\{\log(c_t) + \frac{1}{1+\rho}E_t(a \ln A_{t+1} + b)\}$$

subject to

$$A_{t+1} = (1+r)A_t - c_t$$

FOC w.r.t. c_t gives

$$0 = \frac{1}{c_t} + \frac{1}{1+\rho}E_t(\frac{-a}{A_{t+1}}) \tag{1}$$

FOC w.r.t. A_t gives

$$\frac{a}{A_t} = \frac{1+r}{1+\rho}E_t(\frac{-a}{A_{t+1}})$$

Combining the last two equations gives

$$\frac{a}{A_t} = \frac{1+r}{c_t} \quad \Rightarrow \quad a = \frac{(1+r)A_t}{c_t}$$

Using this equality and the Budget Constraint together with the equation (1)

$$\frac{1}{c_t} = \frac{a}{(1+\rho)A_{t+1}} = \frac{(1+r)A_t/c_t}{(1+\rho)[(1+r)A_t - c_t]}$$

finds that

$$(1+r)A_t = (1+\rho)[(1+r)A_t - c_t]$$

rearranging the equation

$$(1+r)A_t - (1+\rho)(1+r)A_t = -(1+\rho)c_t$$

finally

$$-\rho(1+r)A_t = -(1+\rho)c_t$$

If we further use that $r = \rho$, it finds that

$$c_t = rA_t$$

just like we find for the quadratic utility case

Interpretation (though it was not asked): Since there in no uncertainty for the future labor incomes, households do not show prudent behaviour and consumption shows a flat pattern

Q: Dynamic Programming: Consumption Theory with CRRA utility

Consider an individual with an infinite horizon and CRRA preferences

$$\sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\gamma}}{1-\gamma} \tag{1}$$

where $\gamma \geq 1$ is the inverse of the Intertemporal elasticity of substitution. The individual faces a known deterministic income stream$\{y_{t+j}\}_{j=0}^{\infty}$ and saves/borrows at the interest rate r.

a) Write down the Bellman Equation and derive the Euler equation.

Solution: Remember that when there is no uncertainty for future incomes, we define total wealth of consumers as

$$W_t = (1+r)A_t + \sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} = (1+r)(A_t + H_t)$$

so the Bellman Equation is

$$V_t(W_t) = \max_{c_t}\{u(c_t) + \beta V_{t+1}(W_{t+1})\}$$

which is subject to

$$W_{t+1} = (1+r)(W_t - c_t)$$

Taking the first order conditions w.r.t. W_t

$$V_t'(W_t) = \beta(1+r)V_{t+1}'(W_{t+1}) \tag{2}$$

Using (1) and taking the first order conditions w.r.t. c_t

$$c_t^{-\gamma} = \beta(1+r)V_{t+1}'(W_{t+1}) \tag{3}$$

Combining (2) and (3) find the Euler Equation

$$c_t^{-\gamma} = \beta(1+r)c_{t+1}^{-\gamma} \tag{4}$$

b) What is the optimal path of consumption when $\beta(1+r) = 1$?

Solution: (4) reduces to

$$c_t = c_{t+1}$$

Using the intertemporal budget constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_{t+i} = (1+r)A_t + \sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} = (1+r)(A_t + H_t)$$

which reduces to

$$\frac{1+r}{r}c_t = (1+r)(A_t + H_t)$$

and finds

$$c_t = r(A_t + H_t)$$

Q: Proportional Earnings Tax

Consider the permanent income hypothesis in the deterministic case with quadratic utility and $\beta(1+r) = 1$. Suppose the government, unexpectedly, at date t introduces a proportional tax τ on earnings, meaning that it taxes away a fraction τ of earnings.

a) Imagine this tax is transitory, meaning that it is effective only at date t. Determine by how much consumption falls compared to the no tax case.

Solution: In this case W is defined as

$$W_t = (1+r)A_t + \sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} - \tau y_t$$

But the following result of the Bellman equation does not change

$$c_t = c_{t+1}$$

Using the intertemporal budget constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_{t+i} = (1+r)A_t + \sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} - \tau y_t$$

which reduces to

$$c_t = rA_t + \frac{r}{1+r}[\sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} - \tau y_t]$$

b) Imagine this tax is permanent, meaning that it is effective at every date $t+j$ with $j \geqslant 0$. Determine by how much consumption falls compared to the no tax case.

Solution:

$$c_t = rA_t + \frac{r}{1+r}[\sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} - \sum_{i=0}^{\infty}(\frac{1}{1+r})^i \tau y_{t+i}]$$

Q: Certainty Equivalence

Consider the standard consumption problem with stochastic labor income y_t. The consumer's maximization problem is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

given an initial level of assets A_0 and subject to the usual no-Ponzi game condition and the dynamic budget constraint

$$A_{t+1} = RA_t + y_t - c_t$$

where $R = 1 + r$ is the (risk-free and constant) return on asset holdings. Suppose the utility function is quadratic.

$$u(c) = -1/2\theta(\gamma - c)^2, \quad \theta > 0$$

a) Suppose $y_t = y$. Write down the Bellman equation and derive the Euler equation for consumption.

Solution: Remember that when there is no uncertainty for future incomes, we define total wealth of consumers as

$$W_t = (1 + r)A_t + \sum_{i=0}^{\infty} (\frac{1}{1 + r})^i y = (1 + r)(A_t + H_t)$$

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so the Bellman Equation is

$$V_t(W_t) = \max_{c_t}\{u(c_t) + \beta V_{t+1}(W_{t+1})\}$$

which is subject to

$$W_{t+1} = R(W_t - c_t)$$

Taking the first order conditions w.r.t. W_t

$$V_t'(W_t) = \beta R V_{t+1}'(W_{t+1}) \tag{2}$$

Using (1) and taking the first order conditions w.r.t. c_t

$$u'(c_t) = \beta R V_{t+1}'(W_{t+1}) \tag{3}$$

combining (2) and (3)

$$u'(c_t) = \beta R u'(c_{t+1})$$

Substituting quadratic utility

$$c_t = \beta R c_{t+1} + (1 - \beta R)\gamma \tag{4}$$

b) Assume that $\beta R = 1$. What is the optimal policy for consumption (c as a function of the state variable A)? [Hint: Use the method of undetermined coefficients: guess a functional form for the policy rule, verify that it satisfies the differential equation for all values of the state variable and calculate its coefficients].

Solution: Using (4) in terms of the policy rule, $c_t = c(A_t)$

$$c(A_t) = \beta R c(RA_t + y - c(A_t)) + (1 - \beta R)\gamma \tag{45}$$

Since the Euler equation is linear, we guess that the policy rule may be linear as well, $c(A_t) = aA_t + b$, so that (5) becomes

$$\begin{aligned} aA_t + b &= a(RA_t + y - aA_t - b) + b \\ A_t &= (R - a)A_t + y - b \end{aligned}$$

This is satisfied for all A_t iff $R - a = 1 \Leftrightarrow a = R - 1 = r$ and $b = y$, so that the optimal policy is to consume the annuity value of assets plus labor income

$$c = rA + y$$

c) Show that the condition $\beta R = 1$ implies that consumption is constant over time. Is there any other reason in this model why consumption might be constant over time?

Solution: That consumption is constant over time follows immediately from the Euler equation in part a, which -with $\beta R = 1$- says $c_{t+1} = c_t$. In this model, consumption can also be constant over time if assets high enough so that c_t becomes feasible, in which case it is optimal to set $c_t = \gamma$ (in this utility function γ is the satiation point. Even if it is feasible, the consumer does not like to consume more than that)

d) Now let y_t be an i.i.d. random variable. What is/are the state variable(s)? Why?

Solution: Labor income y_t is not a separate state variable because it is i.i.d. so knowing income today does not tell you anything about the state of the world tomorrow

e) Write down the Bellman equation and derive the stochastic Euler equation for consumption. Again assume $\beta R = 1$. What is the optimal policy in this case?

Solution: Stochastic Euler equation

$$\begin{aligned} u'(c_t) &= \beta R E_t[u'(c_{t+1})] \\ c_t &= E_t c_{t+1} \\ c(A_t) &= E_t[\beta R c(RA_t + y_{t+1} - c(A_t)] \end{aligned}$$

The Euler equation is the same as with deterministic labor income, except that c_{t+1} is replaced by $E_t c_{t+1}$. Try a linear policy rule again, $c(A) = aA + b$

$$\begin{aligned} aA_t + b &= a(RA_t + E[y_{t+1}] - aA_t - b) + b \\ A_t &= (R - a)A_t + E[y_{t+1}] - b \end{aligned}$$

This equation is satisfied for all A_t if $a = R - 1 = r$ and $b = E_t[y_{t+1}] = E[y]$, where the last equality follows from y_t i.i.d. The optimal policy under uncertain labor income is given by

$$c = rA + E[y]$$

f) In what sense is the stochastic model equivalent to the deterministic model and in what sense is it not?

Solution: The consumer behaves as if labor income were certain to be $E[y]$. Therefore, the policy rule is very similar, just y is replaced by its expectation. This is called certainty equivalence. The stochastic model is not equivalent to the deterministic one in the sense that realizations of labor income may deviate from its mean and therefore consumption is no longer constant.

Q: Doctorate: Stochastic Asset Returns

Consider the intertemporal consumption allocation decision of a consumer who does not earn any labor income but lives of the return R she gets on her assets A. The dynamic budget constraint is given by

$$A_{t+1} = R_{t+1}(A_t - c_t)$$

Asset returns R_t are an i.i.d. random variable. Because the consumer cannot perfectly forecast the returns on her assets in the future, she solves a stochastic optimization problem, maximizing the expected net present value of utility over consumption

$$\sum_{t=0}^{\infty} \beta^t E_t\left(\frac{c^{1-\theta}}{1-\theta}\right) \tag{1}$$

a) What are the coefficient of relative risk aversion and the coefficient of absolute risk aversion for these preferences? What does that mean?

Solution:

$$u'(c) = c^{-\theta} \qquad\qquad u''(c) = -\theta c^{-\theta-1}$$

$$RRA = -\frac{u''(c)c}{u'(c)} = \theta \qquad\qquad ARA = -\frac{u''(c)}{u'(c)} = \frac{\theta}{c}$$

RRA is constant (CRRA form). This means that a very rich consumer and a very poor one have identical preferences over lotteries over the same relative changes in their consumption levels. Therefore, rich consumers are willing to take on gambles that are larger in absolute terms and CARA is decreasing in consumption levels.

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b) Write down the Bellman equation and derive the Euler equation for consumption.

Solution: Bellman equation

$$V_t(A_t) = \max_{c_t}\{u(c_t) + \beta E_t V_{t+1}(A_{t+1})\}$$

subject to

$$A_{t+1} = R_{t+1}(A_t - c_t)$$

Taking the first order conditions w.r.t. A_t (the Envelope Condition)

$$V_t'(A_t) = \beta E_t[R_{t+1}V_{t+1}'(A_{t+1})] \tag{2}$$

Using (1) and taking the first order conditions w.r.t. c_t

$$u'(c_t) = \beta E_t[R_{t+1}V_{t+1}'(A_{t+1})] \tag{3}$$

combining (2)and (3)

$$u'(c_t) = \beta E_t[R_{t+1}u'(c_{t+1})]$$

Substituting CRRA utility

$$c_t^{-\theta} = \beta E_t[R_{t+1}c_{t+1}^{-\theta}] \tag{4}$$

c) Solve for the policy rule.

Solution: Using the Merton's rule

$$aA_t^{\alpha} + b = \max_{c_t}\{u(c_t) + \beta E_t(aA_{t+1}^{\alpha} + b)\}$$

subject to

$$A_{t+1} = R_{t+1}(A_t - c_t)$$

Taking the first order conditions w.r.t. A_t (the Envelope Condition)

$$A_t^{\alpha-1} = \beta E_t[R_{t+1}A_{t+1}^{\alpha-1}] \tag{2}$$

Using (1) and taking the first order conditions w.r.t. c_t

$$c_t^{-\theta} = \beta E_t[R_{t+1}a\alpha A_{t+1}^{\alpha-1}] \tag{3}$$

(2) and (3) implies that

$$c_t^{-\theta} = a\alpha A_t^{\alpha-1} \quad \text{and} \quad \alpha = 1-\theta$$

this equation only tells us that c_t is linear in A_t.

Using (2) again

$$A_t^{\alpha-1} = \beta E_t[R_{t+1}^{\alpha}(A_t - c_t)^{\alpha-1}]$$

Taking the variables known at time t out of expectation

$$A_t^{\alpha-1} = \beta(A_t - c_t)^{\alpha-1} E_t[R_{t+1}^{\alpha}]$$

reaaranging the equation

$$A_t = \beta^{1/(\alpha-1)}(A_t - c_t)E_t[R_{t+1}^{\alpha}]^{1/(\alpha-1)}$$

we find

$$c_t = A_t[1 - \beta^{-1/(\alpha-1)} E_t[R_{t+1}^{\alpha}]^{-1/(\alpha-1)}]$$

using the value of α

$$c_t = A_t[1 - \beta^{1/\theta} E_t[R_{t+1}^{1-\theta}]^{1/\theta)}]$$

Q: Doctorate: Log-normal Approximation

In the model of the previous question, suppose that the interest is constant. Suppose also that income is i.i.d. with some (unspecified) distribution that guarantees that the distribution of c_{t+1} conditional on information at time t is log-normal. Further, assume that the conditional variance of $\log c_{t+1}$ is constant over time and equal to σ^2. The consumer has a CRRA equal to θ.

a) Show that marginal utility is log-normal as well. What are the mean and variance of $\log u'(c_{t+1})$ conditional on information at time t?

Solution: The question tells you that $\log c_{t+1} \sim N(E_t[\log c_{t+1}],\, \sigma^2)$.
Since $\log u'(c_{t+1})$=$\log c_{t+1}^{-\theta} = -\theta \log c_{t+1}$, we get that $\log u'(c_{t+1}) \sim N(-\theta E_t[\log c_{t+1}], \theta^2\sigma^2)$

b) Show that $\log c_t$ follows a random walk (with drift). [Hint: Substitute the functional form for the utility function into the Euler equation and take logaritms on both sides. Evaluate the expectation on the right-hand side using the fact that c_{t+1} is log-normal and $Var_t[\log c_{t+1}] = \sigma^2$]

Solution: The Euler equation is still the same (the state is now cash-in-hand, $X_t = A_t + y_t$)

$$u'(c_t) = \beta R E_t[u'(c_{t+1})] \quad \Rightarrow \quad c_t^{-\theta} = \beta R E_t[c_{t+1}^{-\theta}]$$

Using the expression for the expectation of a log-normal variable, we get

$$c_t^{-\theta} = \beta R E_t[c_{t+1}^{-\theta}] = \beta R \exp(-\theta E_t[\log c_{t+1}] + \theta^2\sigma^2/2)$$

and taking logaritms on both sides of the equation yields

$$E_t[\Delta \log c_{t+1}] = \frac{\theta\sigma^2}{2} + \frac{\log \beta R}{\theta}$$

c) How do changes in interest rates and the volatility of consumption affect consumption growth. Explain the economic intuition for those results.

Solution: A higher volatility increases expected consumption growth or -in other words decreases consumption today. This is precautionary savings. How strong this effect is depends on θ, which is a measure of prudence (as well as risk aversion and intertemporal substitutability, but here prudence is what matters). Higher interest rates (higher R) increase consumption growth because of a substitu tion effect: saving becomes more attractive so the consumer postpones more consumption for later.

Q: Doctorate: Different Stochastic Processes for Income

Consider the consumption-savings choice of an agent whose dynamic budget constraint is given by

$$A_{t+1} = R(A_t + y_t - c_t)$$

Asset returns R are fixed, but labor income y_t is stochastic.

1. Suppose income follows an AR(1) process

$$y_t = \rho y_{t-1} + \varepsilon_t$$

   where ε_t is i.i.d. and has mean zero and $\rho < 1$. What is the Euler equation? What variables does the policy rule for consumption depend on?

2. Now suppose income follows an AR(2) process

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t$$

   where $\rho_1 + \rho_2 < 1$. What is the Euler equation? What variables does the policy rule for consumption depend on?

3. Now suppose income follows an MA(1) process

$$y_t = \varepsilon_t + \phi\varepsilon_{t-1}$$

where we assume the consumer observes y_t but not ε_t. What is the Euler equation? What variables does the policy rule for consumption depend on?

Solution: The Euler equation is always the same.

$$u'(c_t) = \beta R E_t[u'(c_{t+1})]$$

This is a fairly general result: as long as the stochastic process for income does not depend on variables of the model (e.g. A_t or c_t), i.e. as long as income is exogenous, it does not affect the Euler equation. In part 1, the state is A_t and y_t (separately). y_{t-1} is not included because y_t is sufficient to predict the future income. In part 2, A_t, y_t and y_{t-1} and in part 3, A_t and all past value of y, i.e. y_t, y_{t-1}, y_{t-2},... (You could figure that out the last result by asking yourself which variables help predict income tomorrow. To predict y_{t+1}, we would like to know ε_t (and ε_{t+1} but since that innovation is only realized in the next period it is hopeless). We cannot observe ε_t, but because $\varepsilon_t = y_t - \phi\varepsilon_{t-1}$, y_t must be in the state. And since $\varepsilon_{t-1} = y_{t-1} - \phi\varepsilon_{t-2}$, y_{t-1} must be in the state too. Repeating this argument you will see that all past values of income help predict income tomorrow)

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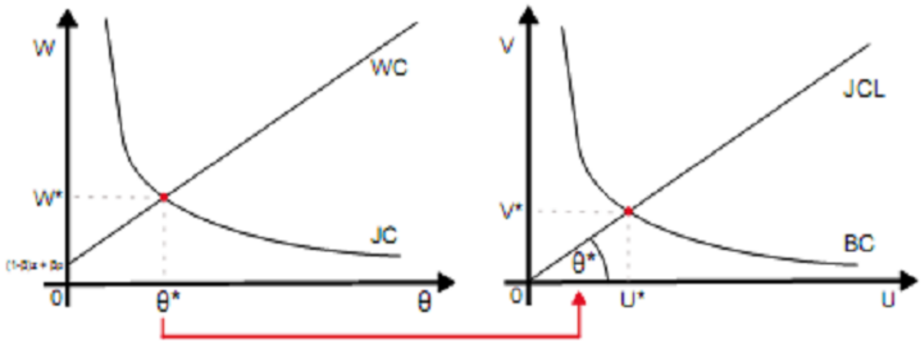
Q: The Search and Matching Model (p)

Recall the three equations (Beveridge Curve, Job Creation Condition and Wage Condition) for the equilibrium conditions of the Search and Matching Model

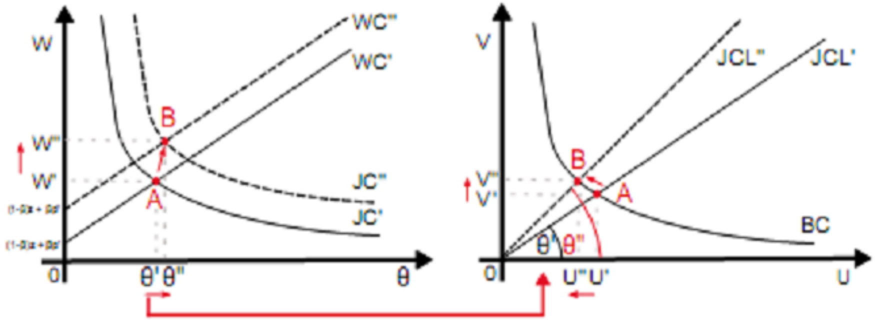
$$BC : u = \frac{\lambda}{\lambda + \theta q(\theta)} \qquad JC : p - w = \frac{(r+\lambda)pc}{q(\theta)} \qquad WC : w = (1-\beta)z + \beta p(1 + c\theta)$$

Analyze the effect of an increase in productivity (from p' to p'') graphically on w, u, v. Also give an economic interpretation when you move the graphs



$$BC: u = \frac{\lambda}{\lambda + \theta q(\theta)} \qquad JC: p - w = \frac{(r+\lambda)pc}{q(\theta)} \qquad WC: w = (1-\beta)z + \beta p(1 + c\theta)$$

Solution:



· The rise in productivity (p) shifts Wage Equation (WC) upwards because the more productive workers demand higher wage for any level of θ

· The rise in productivity (p) also shifts Job Creation Equation (JC) rightward because for any level of w, more productive workers give incentive to employers for opening new job positions (vacancies). So v increases, and so does $\theta = v/u$

· As a result: the wage w rises (as it is seen on the left figure)

· Labor market tightness (θ) may rise or not depending on the value of β and c (depending on the intersection of new JC and WC curves)

If θ rises: JCL rotates counterclockwise:Vacancies (v) rise, unemployment (u) falls

If θ declines: JCL rotates clockwise:Vacancies (v) fall, unemployment (u) rises

Q: The Search and Matching Model (λ)

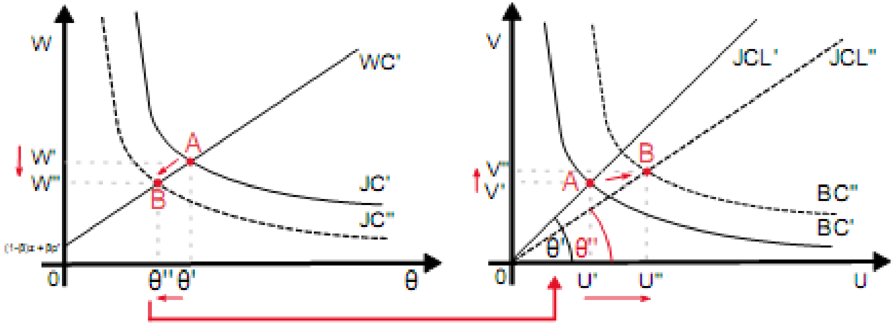
Recall the three equations (Beveridge Curve, Job Creation Condition and Wage Condition) for the equilibrium conditions of the Search and Matching Model

$$BC : u = \frac{\lambda}{\lambda + \theta q(\theta)} \qquad JC : p - w = \frac{(r+\lambda)pc}{q(\theta)} \qquad WC : w = (1-\beta)z + \beta p(1 + c\theta)$$

Analyze the effect of an increase in the job separation rate (λ) graphically on w, u, v. Also given an economic interpretation when you move the line and curves

$$BC: u = \frac{\lambda}{\lambda + \theta q(\theta)} \qquad JC: p - w = \frac{(r+\lambda)pc}{q(\theta)} \qquad WC: w = (1-\beta)z + \beta p(1 + c\theta)$$

Solution:



· The increase in the job separation rate (λ) lowers the value of occupied jobs for employers. Hence, for any level of w, they open less vacancies, v falls, and so does θ ($= v/u$). Therefore, Job Creation Equation (JC) shifts letfwards

· Lower θ rotates JCL curve clockwise, as it is seen on the right figure. However, BC curve is also affected from the change in the job separation rate (λ)

· The higher the λ, the more unemployment model produces for any number of v. Hence, BC curve shifts rightward

· At the equilibrium point B, we have more unemployment for sure. Whether v increases or not depends on the model parameters

Q: Eliminating unemployment

"Eliminating unemployment is a simple matter of reducing workers' bargaining power". Discuss this claim using the Search-Matching model. Then analyze the outcome of the model when firms have no bargaining power. Contrast the two cases.

Solution: In the Search-Matching model equilibrium unemployment is given by

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

Hence, the only way to eliminate unemployment is etiher set $\lambda = 0$, or when $\lambda > 0$ it is to have $\theta \to \infty$, which makes it infinitely easy to find a job.

To see the effect of workers' bargaining power on the equlibrium of the model let's get rid of w by combining job creation and wage setting conditions:

$$w = p - \frac{(r+\lambda)pc}{q(\theta)} = (1-\beta)z + \beta p(1 + c\theta)$$

If we reduce workers' bargaining power to $\beta = 0$,

$$w = p - \frac{(r+\lambda)pc}{q(\theta)} = z$$

Therefore, the equilibrium tightness parameter is such that

$$q(\theta) = \frac{(r+\lambda)}{p-z} pc$$

Hence θ is finite, and unemployment is not eliminated.

The opposite case, where $\beta = 1$ implies that

$$c\theta = -\frac{(r+\lambda)c}{q(\theta)}$$

However, it can never be satisfied since the term on the left-hand side is positive and the one on the right-hand side is negative. The intuition is quite simple: since the worker appropriates all the rent, firms cannot recover their searching costs. Consequently no firm will enter the market and we have full unemployment.

Hence, when there are search frictions in the labour market unemployment is not eliminated.

Q: SM-Minimum Wages

Consider a standard Search-Matching model but all jobs have the same productivity P and firms post vacancies at a flow cost equal to C. (Remember that in the class we assumed that firms post vacancies at a flow cost equal to pc.) The positive unemployment income of workers is equal to z.

a) Derive the Job Creation and the Wage Curves.

Solution: This question asks to redo the calculations explained during the lectures. We briefly repeat them here.

Job Creation Curve: The asset equations for Vacancies and Filled Jobs are

$$\begin{aligned} rV &= -c + q(\theta)(J - V) \\ rJ &= p - w - \lambda J \end{aligned}$$

Free-Entry implies that in steady-state $V = 0$, therefore

$$J = \frac{p - w}{r + \lambda} = \frac{c}{q(\theta)}$$

Wage Curve: The asset equations for Unemployed and Employed workers are

$$\begin{aligned} rU &= z + \theta q(\theta)(W - U) \\ rW &= w + \lambda(U - W) \end{aligned}$$

The second asset equation can be rewritten as follows

$$w = (r + \lambda)(W - U) + rU$$

To obtain the value of $(W - U)$ in terms of the model's parameters, we use the Nash-bargaining solution ($\beta J = (1 - \beta)(W - U)$)

$$rU = z + \theta q(\theta)(W - U) = z + \theta q(\theta)\frac{\beta}{1-\beta}J = z + \theta\frac{\beta}{1-\beta}c$$

Finally reinsert this solution into (6) to obtain

$$\begin{aligned} w &= (r + \lambda)(W - U) - rU \\ &= \frac{\beta}{1-\beta}(p - w) + z + \theta\frac{\beta}{1-\beta}c \end{aligned}$$

Simplifying this expression yields

$$w = (1 - \beta)z + \beta p + \beta c\theta$$

b) The government imposes a minimum wage w_m on firms. Obviously, If w_m is too low it has no effect and if it is too high, firms do not open vacancies. Characterize these two bounds $[\underline{\mathrm{w}}, \bar{w}]$

What is the impact of the minimum wage on θ and u?

Solution: If the minimum wage is below the market wage, then the regulation is irrelevant. Thus the lower bound is $\underline{\mathrm{w}} = w = (1-\beta)z + \beta p + \beta c\theta$

If the minimum wage is above the job's output, firms have no incentives to post vacancies since they cannot recover their searching costs. Thus the upper bound is $\bar{w} = p$.

For any minimum wage between these two bounds, the equilibrium is well defined and differs from the free-market equilibrium. Since the wage is fixed exogenously, we do not need to consider the wage curve. The equilibrium tightness is given by the JC condition, so that

$$p - w_m = \frac{(r+\lambda)c}{q(\theta)}$$

When w_m is above the market wage, it must be the case that $q(\theta)$ increases. Since $q'(\theta) < 0$, this implies that θ decreases. In other words, there is less vacancies per unemployed worker.

To obtain the impact on the equilibrium rate of unemployment, we use the Beveridge curve

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

We know that θ decreases and since $\partial(\theta q(\theta))/\partial\theta > 0$, we can deduce that the unemployment rate increases

c) Consider once again that the cost of posting a vacancy is proportional to P, so that $C = cP$ We also assume that the flow income of workers is proportional to P, so that $Z = zP$. What is the new expression of the equilibrium wage?

Solution: We replace the new expressions in the WC to obtain

$$\begin{aligned} w &= \beta P + \beta\theta c P + (1-\beta) z P \\ &= P[\beta + \beta\theta c + (1-\beta) z] \end{aligned}$$

Therefore the wage is given by a constant factor times the job's output.

d) What is the impact of a change in P on wages, θ and unemployment ?

Solution: Take the expression for w from part b and isert it into the JC condition

$$\begin{aligned}
\frac{P-w}{r+\lambda} &= \frac{cP}{q(\theta)} \\
\frac{P[1-\beta+\beta\theta c+(1-\beta)z]}{r+\lambda} &= \frac{cP}{q(\theta)} \\
&\Rightarrow \frac{1-\beta+\beta\theta c+(1-\beta)z}{r+\lambda} = \frac{c}{q(\theta)}
\end{aligned}$$

P has no impact on model variables

e) Assume that there is technological progress so that $P(t)$ is an increasing function of time. What happens as t goes to infinity if the minimum wage grows at a slower rate than $P(t)$? What happens as t goes to infinity if the minimum wage grows at a faster rate than $P(t)$?

Solution: P is a measure of job's productivity. Thus one can think of changes in P as measuring technological progress.

If the minimum wage grows at a slower rate than technological progress, it is obvious that after a sufficient length of time $w(t) = P(t)[\beta + \beta\theta c + (1 - \beta)z] > w_m(t)$ so that the market wage will eventually exceeds the minimum wage.

On the contrary, if the minimum wage grows at a faster rate than technological progress, it will always be true that $\lim_{t\to\infty} w_m(t) > \lim_{t\to\infty} P(t)$. Then the equilibrium will collapse and there will be full-unemployment.

To conclude, for the minimum wage to be consistent with steady state and growth, it must increase at the same rate than technological progress.