PERMANENT INCOME AND OPTIMAL CONSUMPTION

On the previous notes we saw how permanent income hypothesis can solve the Consumption Puzzle. Now we use this hypothesis, together with assumption of rational expectations, to analyze the optimal consumption decision of people

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Some Background Information: Utility Functions

- Relative Risk Aversion (RRA)$= -\dfrac{\dfrac{\partial u^{'}(c)}{u^{'}(c)}}{\dfrac{\partial c}{c}} = -\dfrac{u''(c)c}{u'(c)}$

- Absolute Risk Aversion (ARA)$= -\dfrac{\dfrac{\partial u^{'}(c)}{u^{'}(c)}}{\partial c} = -\dfrac{u''(c)}{u'(c)}$

  – If the coefficient of relative risk aversion (RRA) is constant for a utility function, then we say it is CRRA type. If coefficient of absolute risk aversion (ARA) is constant, then it is CARA type

- Assume the functional form: $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta} \quad \theta > 0$

$$u'(c) = (1-\theta)\frac{c^{-\theta}}{1-\theta} = c^{-\theta} \qquad\qquad u''(c) = -\theta c^{-\theta-1}$$

$$-\frac{u''(c)c}{u'(c)} = \theta \qquad\qquad -\frac{u''(c)}{u'(c)} = \frac{\theta}{c}$$

$\Rightarrow$ The isoelastic utility function has CRRA form

Also $u''(c) = -\theta c^{-\theta-1} < 0$ and $u'''(c) = \theta(1+\theta)c^{-\theta-2} > 0$

(Soon we will analze the importance of third derivative)

- Assume the functional form: $u(c) = \log c$

$$u'(c) = \frac{1}{c} \qquad u''(c) = -\frac{1}{c^2}$$

$$-\frac{u''(c)c}{u'(c)} = 1 \qquad -\frac{u''(c)}{u'(c)} = \frac{1}{c}$$

$\Rightarrow$ The logarithmic utility function has CRRA form

Also $u''(c) = -\frac{1}{c^2} < 0$ and $u'''(c) = \frac{2}{c^3} > 0$

The logarithmic utility function is isoelastic utility function with $\theta = 1$

- Assume the functional form: $u(c) = -(1/\theta)e^{-\theta c} \qquad \theta > 0$ and $\theta \neq 0$

$$u^{'}(c) = -(1/\theta)(-\theta)e^{-\theta c} = e^{-\theta c} \qquad u^{''}(c) = -\theta e^{-\theta c}$$

$$-\frac{u''(c)c}{u'(c)} = \theta c \qquad -\frac{u'(c)}{u''(c)} = \theta$$

$\Rightarrow$ Exponential Utility function has CARA form

Also $u''(c) = -\theta e^{-\theta c} < 0$ and $u'''(c) = \theta^2 e^{-\theta c} > 0$

- Assume the functional form (Quadratic utility): $u(c) = c - \frac{b}{2}c^2$

$$u'(c) = 1 - bc > 0 \qquad u''(c) = -b < 0 \qquad u'''(c) = 0$$

A note on notation

- In the Ramsey model we maximized the utility $U = \int_0^\infty u[c(t)]e^{nt}e^{-\rho t}dt$
  - Now we assume the population size is constant, i.e. $n = 0$
  - We use discrete time and uncertainty for future, and try to maximize

$$U_t = E_t[\sum_{i=0}^{\infty}(\frac{1}{1+\rho})^i u(c_{t+i})]$$

    * When ρ is small, $e^{-\rho t}$ can be written as $(1-\rho)$. And this can approximated also by $1/(1+\rho)$ (you can use $\rho = 0.02$ to see this)

- - * If there is an uncertainty for the state of the nature in the future, and if consumers faces an intertemporal budget constraint, they need to make some forecasts for the future states (for example for their future incomes, y_{t+i}). *Rational-expectations* assumption states that people use all available information to make optimal forecasts about the future. E_t stands for this assumption

    * There is also *Adaptive Expectations*, according to which people form their expectations about the future based on what has happened in the past

$$c_{t+1}^e = c_t^e + \varphi(c_t - c_t^e)$$

Consumer Problem

- For $i = 0, 1, 2, ..., \infty$, consumers maximize the utility function

$$\max_{c_{t+i}} U_t = E_t[\sum_{i=0}^{\infty}(\frac{1}{1+\rho})^i u(c_{t+i})] \tag{1}$$

subject to the series of budget constraints

$$A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i} \tag{2}$$

and the restriction that consumer's debt cannot grow at a rate greater than the financial return r (No-Ponzi Game Condition)

$$\lim_{i\to\infty}(\frac{1}{1+r})^i A_{t+i} \geq 0 \tag{3}$$

  - Optimizing consumers do not leave positive assets when time goes to infinity. As a result, in the solution of the problem, equation (3) holds with strict equality (Transversality Condition)

  – Note: In the continuous time version of this problem we used Hamiltonian to solve it. In this discrete time case, we can either insert equation (2) inside equation (1) for c_{t+i}, or constraint equation (1) with an equation obtained by combining series of budget constraints (2) and use the Lagrangian. I follow the 1st option

- Inserting equation (2) into equation (1)

$$\max_{A_{t+i},\ i=1,..,\infty} U_t = E_t[\sum_{i=0}^{\infty}(\frac{1}{1+\rho})^i u(A_{t+i+1} - (1+r)A_{t+i} - y_{t+i})]$$

taking derivative with respect to A_{t+i+1}

$$(\frac{1}{1+\rho})^i u'(c_{t+i}) + (\frac{1}{1+\rho})^{i+1} u'(c_{t+i+1})[-(1+r)] = 0$$

rearranging the equation

$$E_t u'(c_{t+i}) = \frac{1+r}{1+\rho} E_t u'(c_{t+i+1})$$

since $u'(c_t)$ is known at time t, $E_t u'(c_t) = u'(c_t)$ and we obtain the *Euler Equation:*

$$u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}) \qquad (4^*)$$

- The Euler equation gives the dynamics of marginal utility in any two successive periods

- At the optimum, the agent is indifferent between consuming immediately one unit of the good, with marginal utility $u'(c_t)$, and saving in order to consume $1+r$ units in the next period

- This equation does not find the level of consumption but only can describe the dynamics of consumption from one period to the next given that we *specify a functional form* for $u(c)$.

- In order to find the level of consumption, it is necessary to use consumption dynamics inside the *present value budget constraint* (as we did in the Ramsey Model)

A Note

- Similarity of equation (4): $u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1})$

  (4)

  with its continuous time version: $\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho)$

  (5)

  – Taylor Approximation: $f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x + \ldots$

  – Using the first order approximation for $u'(c_{t+1})$ in equation (4)

$$u'(c_t) = \frac{1+r}{1+\rho}(u'(c_t) + u''(c_t)(c_{t+1} - c_t))$$

which can be simplified as

$$\frac{c_{t+1} - c_t}{c_t} = -\frac{u'(c_t)}{u''(c_t)c_t}(\frac{r - \rho}{1 + r})$$

which is very similar to equation (5) given that $1 + r \simeq 1$

1 - Quadratic Utility

- When we use a quadratic utility function: $u(c) = c - b/2 \cdot c^2$, its derivative has a linear form: $u'(c) = -b \cdot c$, which further implies that

$$E_t u'(c_{t+1}) = u'(E_t(c_{t+1}))$$

- Using this equation in Euler equation

$$u'(E_t(c_{t+1})) = \frac{1+\rho}{1+r} u'(c_t)$$

- If we assume that $r = \rho$, the above equation reduces to

$$E_t c_{t+1} = c_t \tag{7}$$

This is the main implication of the intertemporal choice model with rational expectations and quadratic utility: the best forecast of next period's consumption is the current consumption

- Note 1: In the next slide we derive the present value budget constraint and then use equation (7) to solve consumption (c) in terms of A and future y's. Just be aware that if the utility function is not quadratic, solving c requires a more complicated procedure. In that case we usually refer to a value function and the Bellman Equation

- Note 2: Equation (7) further implies that

$$c_{t+1} = c_t + u_{t+1}$$

- In this case consumption is a martingale, or a random walk (this means changes in the consumption is unpredictable at time t).

2 - Intertemporal (Present Value) Budget Constraint

- We need to compute the consumer's intertemporal budget constraint. Using the budget constraint (2) at time t

$$c_t = (1+r)A_t + y_t - A_{t+1}$$

- Substituting A_{t+1} for the period $t+2$,we get

$$c_t = (1+r)A_t + y_t - \frac{(c_{t+1} - y_{t+1} + A_{t+2})}{1+r}$$

which can be reduced to

$$c_t + \frac{c_{t+1}}{1+r} = y_t + \frac{y_{t+1}}{1+r} + (1+r)A_t - \frac{A_{t+2}}{1+r}$$

- Similarly

$$c_t + \frac{c_{t+1}}{1+r} + \frac{c_{t+2}}{(1+r)^2} = y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + (1+r)A_t - \frac{A_{t+3}}{(1+r)^2}$$

- Hence, in expected terms

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t \qquad (8)$$

That is to say the present value of consumption flows from t up to ∞ is equal to the consumer's total available resources. The later is given by the sum of the initial financial wealth A_t and the present value of future labor incomes from t up to ∞

1 & 2 - Quadratic Utility and Intertemporal Budget Constraint

- With the quadratic utility we obtained $E_t c_{t+1} = c_t$. Substituting this into equation (9) and dividing by $(1+r)$ gives

$$\frac{1}{r}c_t = A_t + \frac{1}{1+r}\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} \equiv A_t + H_t$$

  where H_t is the "human wealth" and is equal to the present value of future expected labor incomes. Thus,

$$c_t = r(A_t + H_t) \equiv y_t^P$$

  That is to say the amount of consumption at t is the annuity value of total life-time wealth: $r(A_t+H_t)$. That return is defined as permanent income, y_t^P

- (In the absence of taxes) disposable income at time t is the following

$$y_t^D = rA_t + y_t$$

- Then we can write that

$$s_t = y_t^D - c_t = y_t^D - y_t^P = (rA_t + y_t) - (rA_t + rH_t) = y_t - rH_t = y_t^T$$

People consume on the basis of their life-time income (y_t^P). If today's income (y_t^D) is more than this amount, it is called transitory income (y_t^T) and is saved (borrowed if y_t^T is negative)

- Saving can also be written as

$$s_t = y_t - \frac{r}{1+r} \sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t y_{t+i} = - \sum_{i=1}^{\infty} (\frac{1}{1+r})^i E_t \Delta y_{t+i}$$

- Consumer save to face expected future declines of labor income ("saving for a rainy day")

- Let's assume the following first-order autoregressive process generating income for consumers

$$y_{t+1} = \lambda y_t + (1-\lambda)\bar{y} + \varepsilon_{t+1} \qquad E_t(\varepsilon_{t+1}) = 0$$

where $0 \leq \lambda \leq 1$ is a parameter and $\bar{y}$ denotes the unconditional mean of income, implying that

$$c_{t+1} = c_t + (\frac{r}{1+r-\lambda})\varepsilon_{t+1}$$

1 $\lambda = 0$. In this case $y_{t+1} = \bar{y} + \varepsilon_{t+1}$. The innovation in current income is purely transitory. Thus it does not affect the level of income in future periods. For an innovation ε_{t+1}, the consumer's human wealth changes by $\varepsilon_{t+1}/(1+r)$. This change in H_{t+1} determines a variation of permanent income, and we get low marginal propensity to consume

$$c_{t+1} = c_t + (\frac{r}{1+r})\varepsilon_{t+1}$$

2 $\lambda = 1$. In this case $y_{t+1} = y_t + \varepsilon_{t+1}$. The innovation in current income is permanent, causing an equal change of all future incomes. The change in human wealth is ε_{t+1}/r and the variation in permanent income and consumption is ε_{t+1}

$$c_{t+1} = c_t + \varepsilon_{t+1}$$

The excess smoothness of consumption

- If the reaction of consumption to unanticipated changes in income is too smooth, this phenomenon is called excess smoothness. The following section deals with the role of a precautionary saving motive in shaping this reaction

The Role of Precautionary Saving - Microeconomic Foundations

- Suppose $r = \rho$ and the Euler equation becomes: $E_t u'(c_{t+1}) = u'(c_t)$
- Suppose further that at time t income is $\bar{y}$ and at time $t+1$ it is either y^A or y^B, where the average of these numbers is $\bar{y}$
- With the quadratic utility function, marginal utility of consumption coincides with the marginal utility of expected consumption. Hence, when $c_t = E_t(c_{t+1})$:

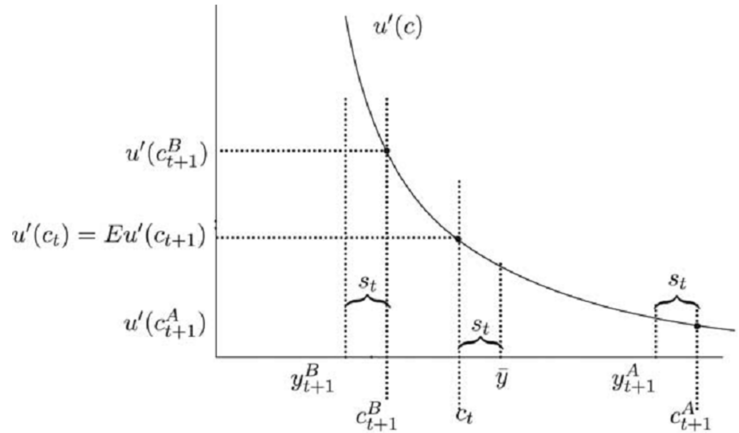
$$E_t u'(c_{t+1}) = u'(E_t(c_{t+1})) = u'(c_t)$$

  - The consumer at time t does not need to save any, as the average of future consumption is just equal to today's consumption, and the Euler Equation is satisfied

- If the utility function is not quadratic; the situation is different. Jensen's inequality states that given a strictly convex function $f(x)$ of a random variable x,

$$E(f(x)) > f(E(x))$$

  – If the utility function is such that the marginal utility is a convex function of consumption (i.e. with risk aversion $u''(c) < 0$ and convex marginal utility $u'''(c) > 0$), then when $c_t = E_t(c_{t+1})$: $E_t u'(c_{t+1}) > u'(E_t(c_{t+1})) = u'(c_t)$. Hence, the Euler equation is satisfied only if $c_t < E_t(c_{t+1})$



- This figure shows that $E_t u'(c_{t+1}) = [u'(y^A_{t+1}) + u'(y^B_{t+1})]/2 > u'(\bar{y}) = u'(c_t)$

- Hence Euler equation is not satisfied at $E_t(c_{t+1}) = c_t$. But if people save at time t, then $E_t u'(c_{t+1}) = [u'(y^A_{t+1} + s_t) + u'(y^B_{t+1} + s_t)]/2 = u'(\bar{y} - s_t) = u'(c_t)$

- This is called a prudent behavior, and defines the situation that consumers reacts to an increase in uncertainty by saving more (precautionary saving). Under uncertainty about future incomes, the consumer fears low-income states and adopts a prudent behavior, saves in the current period in order to increase expected future consumption. This is illustrated with the following figure

- Carroll 1997 and 2001 papers show that precautionary (or buffer-stock) saving is consistent with the empirical facts about consumption to income profile of households

- The definition of the coefficient measuring prudence is similar to that of risk-aversion coefficients: however, the latter is related to the curvature of the utility function (the second derivative), whereas prudence is determined by the curvature of marginal utility (the third derivative)
  - The Coefficient of Relative Prudence$= -\dfrac{\dfrac{\partial u''(c)}{u''(c)}}{\dfrac{\partial c}{c}} = -\dfrac{u'''(c)c}{u''(c)}$
  - The Coefficient of Absolute Prudence$= -\dfrac{\dfrac{\partial u''(c)}{u''(c)}}{\partial c} = -\dfrac{u'''(c)}{u''(c)}$

Non-Quadratic Utility Functions

- Remember the present value budget constraint and the Euler equation

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t \qquad \text{(PVBC)}$$

$$u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}) \qquad \text{(Euler)}$$

- Together with the quadratic utility function, $u(c) = c - b/2 \cdot c^2$, the Euler equation implies that (when $r = \rho$) $E_t c_{t+1} = c_t$. Inserting this into the PVBC enabled us to solve optimal consumption decision of consumers at each point in time in terms of their wealth and income gains (we have even defined an income process and solved the problem completely)

- When the utility function is not quadratic, the Euler equation does not give such straight relationship between consumption at different periods; hence, we need to apply different solution method

- A Note: Quadratic utility function implies that consumer is indifferent between choosing certain future income and or an uncertain future income mean of which is equal to certain one. This property is called *Certainty Equivalence*

Dynamic Programming: Value Function and Bellman Equation
A-Under Certainty on Future Income Flows

- To solve c_t in terms of A_t and H_t, we first define consumer's total wealth as

$$W_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i y_{t+i} + (1+r)A_t = (1+r)(A_t + H_t)$$

where W_t measures the stock of total wealth at the end of period t before consumption c_t occurs. As a result

$$W_{t+i+1} = (1+r)(W_{t+i} - c_{t+i}) \tag{10}$$

Consumer Problem

- For $i = 0, 1, 2, ..., \infty$, consumers maximize the utility function

$$\max_{c_{t+i}} U_t = [\sum_{i=0}^{\infty} (\frac{1}{1+\rho})^i u(c_{t+i})] \tag{1}$$

subject to the series of budget constraints

$$W_{t+i+1} = (1+r)(W_{t+i} - c_{t+i}) \tag{10}$$

and the restriction that the consumer's wealth cannot be negative (the No-Ponzi Game Condition)

$$\lim_{T\to\infty} W_{T+1} \geq 0 \tag{3}$$

- Now we will solve consumption in terms of total wealth: $c_t(W_t)$. In the Dynamic Programming problem W_{t+i} is called as the *state variable* (showing the state, well being, of consumers), and c_{t+i} is the *control variable*

- Consumers get utility $u(\cdot)$ from consuming c. But since they are able to afford c as they have wealth, W, there should be a function, that we call $V(\cdot)$, that would give correspondence of W in terms of utility. Hence the consumer problem at the final period T, instead of $u(c_t(W_t))$, can be written as $V_T(W_T)$, which is called the *Value Function*

- As a result, the consumer problem in period $T-1$ can be written as

$$\max_{c_{T-1}}(u(c_{T-1}) + \frac{1}{1+\rho}V_T(W_T))$$

given W_{T-1}, choosing c_{T-1} determines W_T, which determines the utility for the rest of the periods

- The same procedure can be applied to earlier periods *recursively (backward recursion)*. In general, the problem can be written in terms of *the Bellman equation*:

$$V_t(W_t) = \max_{c_t}\left(u(c_t) + \frac{1}{1+\rho}V_{t+1}(W_{t+1})\right)$$

subject to

$$W_{t+1} = (1+r)(W_t - c_t)$$

  – FOC w.r.t. c_t gives

$$u'(c_t) = \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}) \tag{11}$$

  – FOC w.r.t. W_t gives

$$V'_t(W_t) = \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}) \tag{12}$$

    * Actually there are two additional terms coming from the derivative of c_t w.r.t. W_t, but they cancel out due to the first FOC

  - Combining (11) and (12) gives

$$u'(c_t) = V_t'(W_t)$$

  - Inserting this equation into (11) gives the Euler equation

$$u'(c_t) = \frac{1+r}{1+\rho}u'(c_{t+1})$$

- A Note: Since time is infinite, the iteration of Bellman equation reaches a unique $V(\cdot)$ that is invariant over time

B- Uncertainty on Future Income Flows

- Under uncertainty the solution procedure illustrated above is still appropriate. Let's assume there is uncertainty for the future labor income but not for the interest rate

- This time, instead of W_t, the state variable at time t is the consumer's certain amount of resources at the end of t: $(1+r)A_t+y_t$. As a result the Bellman Equation is

$$V_t[(1+r)A_t+y_t] = \max_{c_t}\{u(c_t)+\frac{1}{1+\rho}E_tV_{t+1}[(1+r)A_{t+1}+y_{t+1}]\}$$

As it is seen, since there is uncertainty for future incomes, there is an expectation term, E_t. This equation is subject to

$$A_{t+1} = (1+r)A_t + y_t - c_t$$

– FOC w.r.t. c_t gives

$$u(c_t) = \frac{1+r}{1+\rho} E_t V'_{t+1}[(1+r)A_{t+1} + y_{t+1}] \tag{13}$$

– FOC w.r.t. A_t gives

$$V'_t(\cdot) = \frac{1+r}{1+\rho} E_t V'_{t+1}(\cdot) \tag{14}$$

– Combining (13) and (14) gives:

$$u'(c_t) = V'_t(\cdot)$$

– Inserting this into the equation (14) gives the Euler equation

$$u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1})$$

Consumption and Financial Returns

- We leave the assumption that the consumer uses a single financial asset with a certain return r. Instead, a consumer can invest his savings in n financial assets with uncertain returns. The chosen portfolio allocation will depend on the characteristics of the consumer's utility function (in particular the degree of risk aversion) and of the distribution of asset returns. We will find the price for assets. This model is the basic version of the consumption-based capital asset pricing model (CCAPM)

- With the new hypothesis of n financial assets with uncertain returns, the consumer's budget constraint must be reformulated accordingly

$$\sum_{j=1}^{n} A_{t+i+1}^{j} = \sum_{j=1}^{n} (1 + r_{t+i+1}^{j}) A_{t+i}^{j} + y_{t+i} - c_{t+i}$$

This equation says that consumers' wealth comes from n financial assets, and after added by labor income and substracted by consumption, can be invested on n different assets as well. The return for asset A^j_{t+i} at at the period $t+i$ is not known at that period. Therefore, we denote it by r^j_{t+i+1} at the period $t+i$

- When Euler equation is solved for each r^j, we get

$$u'(c_t) = \frac{1}{1+\rho} E_t[(1+r^j_{t+1})u'(c_{t+1})]$$

which can be written as

$$1 = E_t[(1+r^j_{t+1})\frac{1}{1+\rho}\frac{u'(c_{t+1})}{u'(c_t)}] \equiv E_t[(1+r^j_{t+1})M_{t+1}]$$

where M_{t+1} is the "stochastic discount factor": the discounted ratio of marginal utilities of consumption in any two subsequent periods

- We know that multiplication of two random variables can be written as

$$1 = E_t[(1 + r^j_{t+1})M_{t+1}] = E_t(1 + r^j_{t+1})E_t(M_{t+1}) + cov_t(r^j_{t+1}, M_{t+1})]$$

then

$$E_t(1 + r^j_{t+1}) = \frac{1}{E_t(M_{t+1})}[1 - cov_t(r^j_{t+1}, M_{t+1})]$$

- For the safe asset (r^0) considered in the previous sections, this reduces to

$$1 + r^0_{t+1} = \frac{1}{E_t(M_{t+1})}$$

subtracting this equation from the previous one gives

$$E_t(r^j_{t+1}) - r^0_{t+1} = -(1 + r^0_{t+1})cov_t(r^j_{t+1}, M_{t+1}) \tag{9}$$

- The stochastic discount factor, $M_{t+1} = \dfrac{1}{1+\rho}\dfrac{u'(c_{t+1})}{u'(c_t)}$, together with (9) gives the main result from the model with risky financial assets: An asset j whose return has a negative covariance with M_{t+1} yields an expected return higher than r^0
  - To see this suppose M_{t+1} is high (i.e. $u'(c_{t+1})$ is high relative to $u'(c_t)$). This means c_{t+1} is low relative to c_t at equilibrium. When the return of an asset j is negatively correlated with M_{t+1}, it gives low return when c_{t+1} is already low. Then the agent holds this asset only if such risk is appropriately compensated by a "premium," given by an expected return higher than the risk-free rate r^0 return (i.e. $E_t(r^j_{t+1}) - r^0_{t+1} > 0$)