C - RAMSEY MODEL

Households

- The model assumes infinitely lived agents
  - This assumption helps us to simplify the world where there are finite lived individuals that are connected through altruism
- It is a closed economy
- Population grows at a constant rate, $\dfrac{\dot{L}}{L} = n$

The Representative Agent's Utility Function

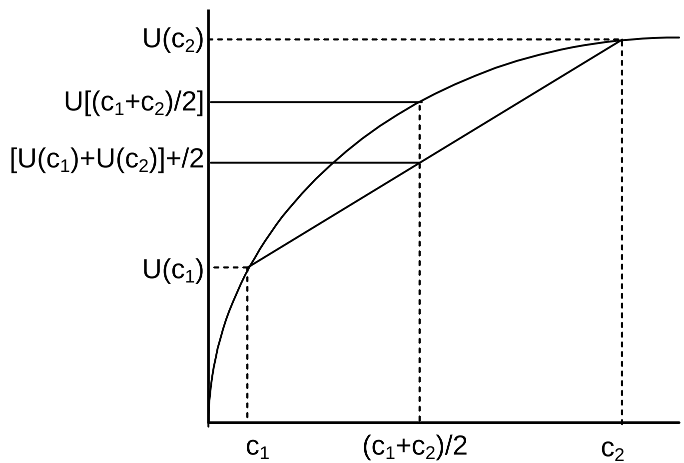
- We consider a household representing the average consumer in the economy. His life time utility function is given as

$$\max U = \int_0^\infty u(c_t)e^{nt}e^{-\rho t}dt \qquad \text{where } \rho > 0 \text{ and } c_t = \frac{C_t}{L_t}$$

$$\simeq u(c_t) + u(c_{t+1})(1+n)(1-\rho) + u(c_{t+2})(1+n)^2(1-\rho)^2 + ...$$

  - $e^{-\rho t}$: rate of time preference for consumption ($\rho > 0$ means that goods are utilized less the later they are received)
  - e^{nt} : This term helps us to consider the fact that there is going to be more consumers in the future
  - $\rho > n$: U is bounded if c is constant over time

- The utility function satisfies: $u'(c) > 0 \quad u''(c) < 0$ (The marginal utility is always positive but decreases with an increase in c)
  - $u''(c) < 0$ implies that instead of consuming c_1 and c_2 and having the average utility of $[U(c_1) + U(c_2)]/2$, households prefer to smooth consumption and consume $(c_1 + c_2)/2$ in both periods



The Representative Agent's Budget Constraint

- Households invest their savings on assets, and the change in total assets in the economy is given as follows

$$\dot{Assets} = \omega_t L_t + r_t Assets_t - C_t$$

- Assets increase with the wage income of households, which is the wage rate times the total labor force in the economy, and with the interest return on their previous assets. Assets decrease with the households' consumption expenditure

- In per capita terms, the above equation can be written as

$$\dot{a}_t = (\frac{\dot{Assets}}{L_t}) = \frac{\dot{Assets} L_t - \dot{L}_t Assets_t}{L_t^2} = \frac{\dot{Assets}}{L_t} - n a_t$$

$$\Rightarrow \dot{a}_t = \omega_t + (r_t - n) a_t - c_t \tag{1}$$

The Representative Agent's Problem

- Households optimize their consumption flow

$$\max \int_0^\infty u(c(t))e^{-(\rho-n)t}dt$$

$$s.t. \;\; \dot{a}_t = \omega_t + r_t a_t - c_t - n a_t$$

(Control variable: c, State variable: a)

Notes on the Solution of the Above Problem

- Notice that the decision for consumption today affects future consumption via wealth of households, a. (Here c is the *control variable* and a is the *state variable*.) This problem is called *Dynamic Optimization in Continuos Time (Optimal Control Approach)*. And we cannot solve this continuous time problem just by taking FOCs

- The details of the solution is in Appendix A3 of: Economic Growth: Barro and Sala-i Martin. It requires writing the problem as *Hamiltonian* and take the FOCs

- The assets in the form of ownership on capital or as loans. The negative assets represent debts. As households can be indebted, we need to use a constraint coming from credit markets

$$\lim_{t\to\infty} a(t) \exp(-\int_0^t [r(v - n]dv) \geqslant 0$$

This is called *Complementary-Slackness condition.* It means that household's debt (negative values of $A(t)$) cannot grow as fast as $r(t)$). It rules out Ponzi schemes for debt

- Otherwise households would have an incentive to borrow for financing current consumption, and use future borrowings to roll over the principal and pay the interest. In this case, the household's debt grows forever

Household's Problem in the Hamiltonian Form

- The Hamiltonian is

$$\mathtt{H} = u(c)e^{(n-\rho)t} + \lambda(\omega_t + r_t a_t - c_t - n a_t)$$

- The first term on is the utility and the term inside the parenthesis is the change in wealth

- λ is the value of an extra unit of asset in units of utility, and called *shadow price of income*

Transversality Condition

- With the Hamiltonian, the Complementary-Slackness condition

$$\boxed{\lim_{t\to\infty} a(t)\exp(-\int_0^t [r(v)-n]dv) \geqslant 0}$$

becomes

$$\boxed{\lim_{t\to\infty} a(t)\exp(-\int_0^t [r(v)-n]dv) = 0} \tag{6}$$

- This is called *Transversality Condition*. It says that if one cannot be indebted as time goes to infinity, it would be suboptimal for households to accumulate positive assets as well. Hence, s/he accumulates zero wealth

The Hamiltonian (again)

$$\mathtt{H} = u(c)e^{(n-\rho)t} + \lambda(\omega_t + r_t a_t - c_t - n a_t)$$

- For the Hamiltonian, (6) can also be written in a way of

$$\lim_{t\to\infty} \lambda(t)a(t) = 0 \tag{3}$$

  - It says that as t goes to infinity, either $a(t)$ should approaches to 0, or if $a(t)$ is positive, then its price, $\lambda(t)$ at time t, should be 0

- The FOCs

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-(\rho-n)t} = \lambda \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial a} = -\dot{\lambda} \quad \Rightarrow \quad (r-n)\lambda = -\dot{\lambda} \tag{2}$$

- Taking time derivative of (1)

$$(1) \quad \Rightarrow \; u''(c)\dot{c}e^{(n-\rho)t} - u'(c)(\rho - n)e^{-(\rho-n)t} = \dot{\lambda}$$

- Substitute this back into (2)

$$(r-n)u'(c)e^{-(\rho-n)t} = u''(c)\dot{c}e^{-(\rho-n)t} - u'(c)(\rho - n)e^{-(\rho-n)t}$$

results in famous *Euler Equation* (where $u''(c) < 0$):

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho) \tag{4}$$

- The Euler equation that is repeated below

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho) \tag{4}$$

tells us that

  - When $r = \rho$, the interest rate is equal to future discount rate, households would select a flat consumption profile with $(\frac{\dot{c}}{c} = 0)$
  - If $r > \rho$, households give up consumption today for more consumption tomorrow $(\frac{\dot{c}}{c} > 0)$
    - The more $-\frac{u''(c)c}{u'(c)}$, the less $\frac{\dot{c}}{c}$ responds to an increase in $r > \rho$

- The term $-\dfrac{u''(c)c}{u'(c)}$ is the elasticity of $u'(c)$ with respect to c and can be written as

$$\frac{u''(c)c}{u'(c)} = \frac{\dfrac{\partial u'(c)}{\partial c}c}{u'(c)} = \frac{\dfrac{\partial u'(c)}{u'(c)}}{\dfrac{\partial c}{c}}$$

It is called *the coefficient of relative risk aversion.* The more elastic utility function, the more risk averse households are; and as a result, the less they would be willing to change their consumption pattern over their life time. In fact, this term is also *1/elasticity of intertemporal substitution*

Example

- Assume the functional form $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta} \quad \theta > 0$

$$u'(c) = (1-\theta)\frac{c^{-\theta}}{1-\theta} = c^{-\theta} \qquad u''(c) = -\theta c^{-\theta-1}$$

hence

$$\Rightarrow \quad \frac{u''(c)c}{u'(c)} = \rho - \frac{-\theta c^{-\theta-1} c}{c^{-\theta}} = -\theta$$

this utility function has constant relative risk aversion (CRRA), which is θ, and constant intertemporal elasticity of substitution, which is $1/\theta$

  - $\theta \uparrow \;\; \Rightarrow$ less willing households are to accept deviations from a uniform c

- $\theta \to 0 \;\; \Rightarrow u(c)$ approaches a linear form in c (to see this use the l'hopital rule), which makes households indifferent to timing of consumption if $r = \rho$
- $\theta \to 1 \;\; \Rightarrow u(c)$ approaches a log-utility form, which we will analyze later

• With this form of utility function, we find that

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$$

- If $\theta \uparrow$, then $\frac{\dot{c}}{c}$ responds less to the gap between r & ρ

- We have found that

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) \tag{5}$$

- This equation only defines consumption path from one period to the next. If we want to define consumption at time t in terms of the consumption at time 0, first we can define the average interest rate between 0 and t as

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(v)dv$$

- Now c can be written as

$$c_t = c_0 \exp\{\frac{1}{\theta}[\bar{r}(t) - \rho]\}t \tag{7}$$

- We have derived the consumption at any point in time w.r.t. each other. Our next question is: 'How we can find the level of consumption at any point in time?'

  – We can use the (life time) *Budget Constraint* . That is we can calculate the life time consumption and equate it to the total (life time) wealth

$$\int_0^\infty c_t e^{-[\bar{r}(t)-n]t} dt = a(0) + \int_0^\infty \omega_t e^{-[\bar{r}(t)-n]t} dt \tag{8}$$

  This equation says that total (life time) consumption (when it is brought to time 0) must be equal to initial wealth plus total (life time) labor income

- Plugging equation (7) into the equation (8) finds

$$c_0 = ([\int_0^\infty \exp[\frac{1-\theta}{\theta}\bar{r}(t) - \frac{\rho}{\theta} + n]dt) \cdot [a(0) + \int_0^\infty \omega_t e^{-[\bar{r}(t)-n]t}dt] \qquad (9)$$

given that we have found c_0 in (9), we can calculate c_t's by using (7)

- (9) shows that (depending on $(1-\theta)/\theta$) an increase in r may increase or decrease c_0. This is because r has two effects. One is what we see in equation (5). An increase in r *substitutes* current consumption to the future. The second effect is the *income effect* that we can realize from (8). Due to this effect an increase in r leads to an increase in life time wealth (as it leads to future value of $a(0)$ to increase) and raises c_t at all dates

- In (9), there is also a wealth effect arising from the fact that consumer's life-time income at time 0 depends also on the interest rate. In this case an increase in r decreases the current value of future labor income gains (the last term in equation (9)) and reduces consumption at all periods.)

  - If $\theta < 1$, substitution effect is large (because consumers care relatively little about consumption smoothing)

  - If $\theta > 1$, income effect is large

  - If $\theta = 1$, which is the case with log utility, the two effects exactly cancel out each other. (The wealth effect is still present though)

Firms

- Use a neoclassical production function

$$Y(t) = [K(t), L(t) \cdot T(t)]$$

  where technology that grows at a rate of x, i.e. $T(t) = e^{Xt} \cdot T(0)$

- Define effective labor as $\hat{L} = L \cdot T(t)$, so that $Y = F(K, \hat{L})$

- When the variables are written in terms of per unit of effective labor: $\hat{y} = Y/(LT)$ and $\hat{k} = K/(LT)$, the production function takes the form $\hat{y} = f(\hat{k})$

- The profits at any point in time

$$\pi = F(K, \hat{L}) - (r+\delta)K - \omega L = \hat{L}f(\hat{k}) - (r+\delta)K - \omega L$$

  – firms pay $r + \delta = R(t)$ as a the rental rate of a unit capital. r, on the other hand, is the net rate of return on capital to their owners since the capital depreciates by δ (notice that bonds and capital are perfectly substitutable)

  – Notice also that although firm's decision is over infinite horizon, this is not a dynamic optimization problem since there is no state variable and firms can rent optimal capital and labor at any point in time. Put it differently, maximizing present value of future profits reduce to a problem of maximizing profit in each period

    * The situation is different with the adjustment cost of capital

- Taking r and ω as given (in the competitive environment), FOCs find

$$\frac{\partial\pi}{\partial K} = 0 = \frac{\partial(\hat{L}f(\hat{k}))}{\partial\hat{k}}\frac{\partial\hat{k}}{\partial K} - (r+\delta) = \hat{L}f'(\hat{k})\frac{1}{LT} - (r+\delta)$$

$$\Rightarrow \quad r = f'(\hat{k}) - \delta \tag{10}$$

$$\frac{\partial\pi}{\partial L} = 0 = Tf(\hat{k}) + \hat{L}\frac{\partial(f(\hat{k}))}{\partial\hat{k}}\frac{\partial\hat{k}}{\partial L} - \omega = Tf(\hat{k}) - \hat{L}f'(\hat{k})\frac{K}{L^2T} - \omega$$

$$\Rightarrow \quad \omega = Tf(\hat{k}) - Tf'(\hat{k})\hat{k}$$

$$\text{remember that } T(t) = e^{Xt} \quad \Rightarrow \quad \omega = [f(\hat{k}) - f'(\hat{k})\hat{k}]e^{xt} \tag{11}$$

Equilibrium

- Closed economy $\Rightarrow \quad a = k$
- Then the equation, $\dot{a} = \omega + ra - c - na$, can be written as

$$\dot{k} = \omega + rk - c - nk$$

- The growth rate of capital per effective worker ($\hat{k} = k/T = ke^{-xt}$)

$$\hat{k}^{\cdot} = k^{\cdot}e^{-xt} - kxe^{-xt} = k^{\cdot}e^{-xt} - \hat{k}x = (\omega + rk - c - nk)e^{-xt} - \hat{k}x$$

- This equation, together with (10) and (11) can be used to derive the resource constraint of the economy (where $\hat{c} = c/\hat{L} = ce^{-xt}$)

$$\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k} \tag{12}$$

- It follows from $\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$ that

$$\frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{c^{\cdot}}{c} - x = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x] \tag{13}$$

- (12), (13) and the TVC in equation (6) implies that

$$\lim_{t\to\infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x] dv\} = 0 \tag{14}$$

  - TVC's explanation: $f'(\hat{k}) - \delta > n + x$ (The steady state rate of return on capital is higher than steady state growth of K)

- The system of equations (12) and (13), the initial condition of $\hat{k}(0)$, and TVC determine the time paths $\hat{c}$ and $\hat{k}$

Steady State

- Steady state growth rates of $\hat{k}$ and $\hat{c}$ are zero. It can be shown that the level of variables, K, C and Y grow at the rate $x + n$

- (12): $\frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x]$

  $\hat{c}^{\cdot} = 0$ if and only if $\hat{c} = 0$ or $f'(\hat{k}) = \delta + \rho + \theta x$

  So at the steady state with positive level of consumption:

  $$f'(\hat{k}^*) = \delta + \rho + \theta x$$

- (13): $\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k}$

  $\hat{k}^{\cdot} = 0$ if and only if $\hat{c} = f(\hat{k}) - (n + \delta + x)\hat{k}$

There are many k values satisfying this equation. But $\hat{c}$ is maximized at the golden rule level of capital:

$$\max \hat{c} = \max(f(\hat{k}) - (n+\delta+x)\hat{k}) \quad \Rightarrow \quad f'(\hat{k}_{gold}) = n+\delta+x$$

- Question: is $\hat{k}^*$ equal to $\hat{k}_{gold}$?
  - Use (14): $\lim_{t\to\infty} \hat{k} \exp\{-\int_0^t [f'(\hat{k}) - \delta - n - x]dv\} = 0$. If capital converges to steady state, then TVC requires the steady-state rate of return, $f'(\hat{k}^*) - \delta$, to exceed $x+n$, the steady-state growth rate of K

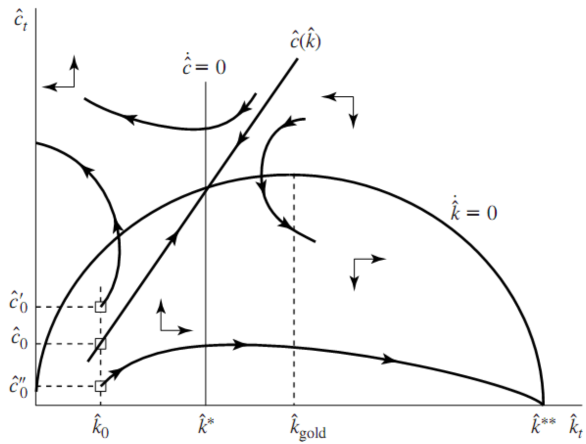
$$f'(\hat{k}^*) - \delta > n + x$$

  - Now we can compare the steady state of capital and $\hat{k}_{gold}$

$$f'(\hat{k}^*) - \delta > n + x \; [= f'(\hat{k}_{gold}) - \delta] \quad \Rightarrow \quad \hat{k}^* < \hat{k}_{gold}$$

Phase Diagram

- (12): $\frac{\hat{c}^{\cdot}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x]$ [$\hat{c}$ is rising for $\hat{k} < \hat{k}^*$ (so the arrows point upward in this region) and falling for $\hat{k} > \hat{k}^*$]
- (13): $\hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k}$ [$\hat{k}$ falls above the curve]

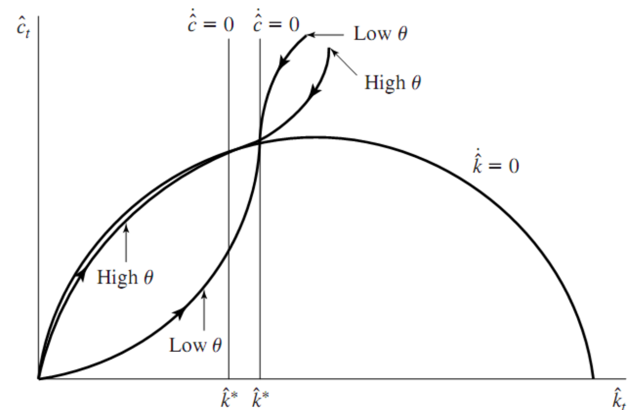


- Results and Notes

  1. Unlike the Solow-Swan model using a constant saving rate, inefficient oversaving ($\hat{k}^* > \hat{k}_{gold}$) cannot occur in this framework
  2. The optimizing households does not sacrifice more of current consumption to reach the maximum amount with k^{gold}
  3. Parameters that describe production function or ρ and θ affect long run level of variables but not the steady-state growth rates
  4. The $\hat{c}^{\cdot} = 0$ and the $\hat{k}^{\cdot} = 0$ lines cross three times so there are three steady states ($\hat{c} = 0$ and $\hat{k} = 0$; $\hat{k}^*$ and $\hat{c}^*$; $\hat{k}^{**} > 0$ and $\hat{c} = 0$). Yet, for any initial positive capital stock, $k(0)$, the only stable equilibria is the one that stays on $\hat{c}(\hat{k})$ is $\hat{k}^*$ and $\hat{c}^*$
  5. The stable arm, $\hat{c}(\hat{k})$, is a *policy function* that expresses the equilibrium $\hat{c}$, the control variable, as a function of $\hat{k}$, the state variable

- Suppose $\hat{k}(0) < \hat{k}^*$ (country starts with little capital stock)
- $\theta \uparrow \;\Rightarrow$ Households have a strong preference for smoothing consumption; Today's consumption increases; Low rate of investment and thus slower convergence; Stable arm lies close to $\hat{k}^{\cdot} = 0$ schedule
- $\theta \downarrow \;\Rightarrow$ Households more willing to postpone c for high rates of return; Stable arm is closer to horizontal axis; Faster transition to the s.s.

The Shape of the Stable Arm:



Behavior of the Saving Rate

- The gross saving rate, $s = 1 - \hat{c}/f(\hat{k})$, depends offsetting impacts from substitution and income effects

- *Substitution Effect:* As country gets richer: $\hat{k} \uparrow \;\Rightarrow\; f'(\hat{k}) \downarrow \Rightarrow r \downarrow \Rightarrow$ An intertemporal substitution reduces, current consumption increases, saving declines

- *Income Effect:* Income per effective worker in a poor economy is far below the long-run income of the economy. Since households like to smooth consumption, this means they consume higher fraction of their income when they are poor, so saving would be low when $\hat{k}$ is low. As $\hat{k}$ rises, consumption falls in relation to current income and saving increases

APPENDIX 1: Alternative Environment

- Separation of households and firms is not crucial to the analysis we have carried out. Below we ignore the markets for capital and labor and solve for the problem of the benevolennt social planner who maximize utility of individuals given the resource constraint in the economy

Benevolent Social Planner

- Maximizing utility of the consumer s.t. the resource constraint of the economy

$$\max \int_0^{\infty} u(c(t)) e^{-(\rho - n)t} dt \qquad s.t. \quad \hat{k}^{\cdot} = f(\hat{k}) - \hat{c} - (n + \delta + x)\hat{k}$$

- Having a state variable, $\hat{k}$, this equation can be solved by Hamiltonian

$$\mathtt{H} = e^{-(\rho-n)t}u(c) + \lambda(f(\hat{k}) - ce^{-xt} - (n+\delta+x)\hat{k})$$

- The first order conditions

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-(\rho-n)t} - \lambda e^{-xt} = 0 \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial \hat{k}} = -\dot{\lambda} \quad \Rightarrow \quad -\dot{\lambda} = \lambda[f'(\hat{k}) - (n+\delta+x)] \tag{2}$$

- Taking first ln and then derivative of (1), solving for $\hat{c}$, and combining it with (2) leads to

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta x],$$

which is equivalent to (13). The solution for the planner will therefore be the same as that for the decentralized economy. Since a benevolent social planner with dictatorial powers will attain a Pareto optimum, the results for the decentralized economy—which coincide with those of the planner—must also be Pareto optimal.

APPENDIX 2: Dynamic Optimization in Discrete Time (Dynamic Programming Approach)

- Assume population growth rate is zero. Social planner maximizes the (expected) utility function of the representative consumer

$$\max_{c_t} U_t = \max_{c_t} E_0 \sum_{t=0}^{\infty} (\frac{1}{1+\rho})^t u(c_t)$$

where $\beta \in (0,1)$ and the flow utility or one-period utility function is strictly increasing $(u\prime(c) > 0)$ and strictly concave $(u''(c) < 0)$, subject to the resource constrain of the economy

$$c_t + k_{t+1} \leq y_t + (1-\delta)k_t$$

and the restriction that the capital stock cannot fall below zero.

$$k_t \geq 0 \quad \forall t$$

- Now we will solve consumption in terms of existing capital stock: $c_t(k_t)$. In the Dynamic Programming problem k_t is called as the *state variable* (showing the state, well being, of society), and c_t is the *control variable.* Finally $c_t(k_t)$ is called the *policy function*

- Consumers get utility $u(\cdot)$ from consuming c. But since they are able to afford c as they have capital, k, there should be a function, that we call $V(\cdot)$, that would give correspondence of k in terms of utility. Hence the consumer problem at the final period T, instead of $u(c_T(k_T))$, can be written as $V_T(k_T)$, which is called the *Value Function*

- As a result, the consumer problem in period $T-1$ can be written as

$$\max_{c_{T-1}}(u(c_{T-1}) + \frac{1}{1+\rho}V_T(k_T))$$

given k_{T-1}, choosing c_{T-1} determines k_T, which determines the utility for the rest of the periods

- The same procedure can be applied to earlier periods *recursively (backward recursion)*. In general, the problem can be written in terms of *the Bellman equation*:

$$V_t(k_t) = \max_{c_t}(u(c_t) + \frac{1}{1+\rho}V_{t+1}(k_{t+1}))$$

subject to

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

where

$$V(k_0) = \sum_{t=0}^{\infty}(\frac{1}{1+\rho})^t u(c_t^*(k_0))$$

where $c_t^*(k_0)$ is the optimal consumption sequence for each possible value of k_0

– FOC w.r.t. c_t gives

$$u'(c_t) = \frac{1+r}{1+\rho} V'_{t+1}(k_{t+1})$$

– FOC w.r.t. W_t gives

$$V'_t(k_t) = \frac{1+r}{1+\rho} V'_{t+1}(k_{t+1})$$

  * Actually there are two additional terms coming from the derivative of c_t w.r.t. k_t, but they cancel out due to the first FOC

– Combining FOCs

$$u'(c_t) = V'_t(k_t)$$

– Inserting this equation into the first FOC, we find the Euler Equation

$$u'(c_t) = \frac{1+r}{1+\rho}u'(c_{t+1})$$

– To see that the final (Euler) equation is analogous to its continuos time version, use the Taylor Approximation:

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x + ..$$

– Using the first order approximation for $u'(c_{t+1})$ in the discrete time Euler Equation

$$u'(c_t) = \frac{1+r}{1+\rho}(u'(c_t) + u''(c_t)(c_{t+1} - c_t))$$

– The final equation can be simplified as

$$\frac{c_{t+1} - c_t}{c_t} = -\frac{u'(c_t)}{u''(c_t)c_t}(\frac{r - \rho}{1 + \rho})$$

which is, given that $1 + \rho \simeq 1$, identical to continuos time version of the Euler Equation, repeated below

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho) \tag{5}$$