Some Definitions

Stochastic version of Samuelson's (1939) Classical Model

$$y_t = c_t + i_t$$
$$c_t = \alpha y_{t-1} + \varepsilon_{ct}$$
$$i_t = \beta(c_t - c_{t-1}) + \varepsilon_{it}$$

- This is a model with three equations, and with three endogenous variables (y_t, c_t, i_t)
- It is a dynamic model (Past variables affect the current variables)
- It is a structural form model. This is because it explains endogenous variables with current realizations of other endogenous variables
  - A reduced form model explains endogenous variables with exogenous ones
    * That is, it explains endogenous variables with their and other endogenous variables' lags (that are called predetermined variables), also current and past values of exogenous variables
  - Reduced form investment equation can be obtained as

$$\begin{aligned} i_t &= \gamma(c_t - c_{t-1}) + \epsilon_{it} = \gamma(\alpha y_{t-1} + \varepsilon_{ct} - c_{t-1}) + \varepsilon_{it} \\ &= \gamma\alpha y_{t-1} - \gamma c_{t-1} + \gamma\varepsilon_{ct} + \varepsilon_{it}, \end{aligned}$$

    which can also be written as

$$i_t = \beta_1 y_{t-1} + \beta_2 c_{t-1} + e_t$$

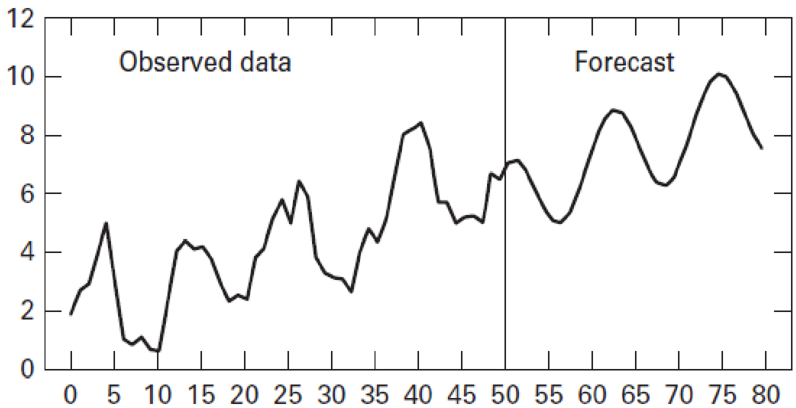
    * Note that the reduced from shock (e_t) is combination of the structural shocks (ε_{ct} & ε_{it})
- Similarly, after some substitutions a reduced-form equation for GDP can be obtained as follows

$$y_t = a y_{t-1} + b y_{t-2} + e_t$$

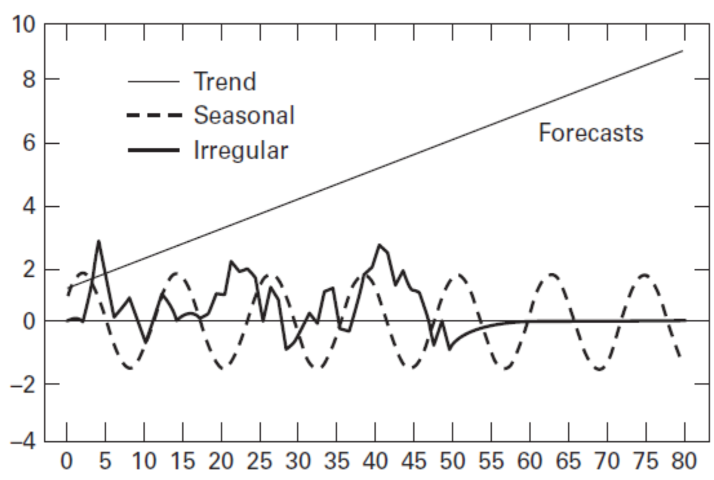
- This is a univariate reduced-form equation; y_t is expressed solely as a function of its own lags and a disturbance term

CHAPTER 1: DIFFERENCE EQUATIONS

- *Difference equation* expresses the value of a variable as a function of its own lagged values, time, and other variables
- *Time-series econometrics* is concerned with the estimation of difference equations containing stochastic components
- Suppose we have the following series as data



- Time series methodology was originally developed to decompose a series into a trend, a seasonal,a cyclical, and an irregular components
  - The trend component represents the long-term behavior of the series
  - The cyclical and seasonal components represent the regular periodic movements
  - The irregular component is stochastic



- Trend: $T_t = 1 + 0.1t$
- Seasonal: $S_t = 1.6sin(t\pi/6)$
- Irregular: $I_t = 0.7I_{t-1} + \varepsilon_t$, where ε_t is random disturbance (such as $\varepsilon_t \sim N(0, \sigma^2)$)

Interpreting the Random Term

$$I_t = 0.7I_{t-1} + \varepsilon_t$$

where 0.7 is the degree of autocorrelation

- Substituting for the lags of I_t

$$\begin{aligned} I_t &= 0.7(0.7I_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= 0.7^2 I_{t-2} + \varepsilon_t + 0.7\varepsilon_{t-1} \\ &= 0.7^2(0.7I_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + 0.7\varepsilon_{t-1} \\ &= 0.7^3 I_{t-3} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.7^2\varepsilon_{t-2} \end{aligned}$$

  showing that the past shocks affect the current state of the variable. (Yet, this effect diminishes over time.)

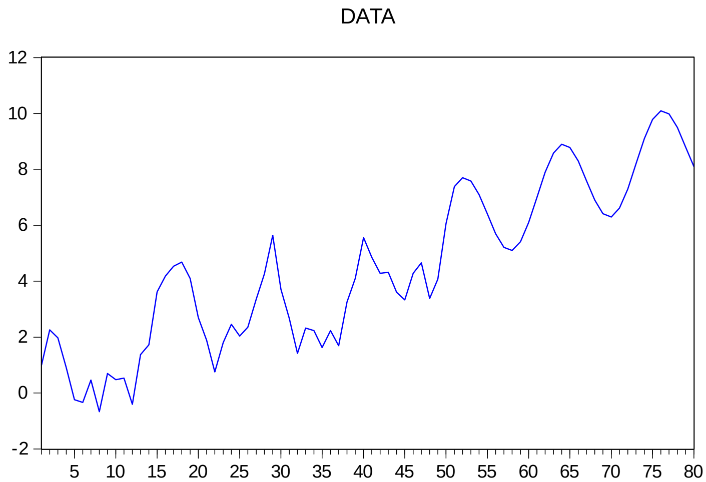
- Writing the last equations at time $t+3$

$$I_{t+3} = 0.7^3 I_t + \varepsilon_{t+3} + 0.7\varepsilon_{t+2} + 0.7^2\varepsilon_{t+1}$$

  showing how the current value of the variable can be used the forecast its future values...

E-views Application

```
wfcreate (wf=income process) u 80
series t=1+0.1*@TREND
!pi = 3.14159
series s=1.6*sin(@TREND*!pi/6)
series e=nrnd
series i=0

smpl @first+1 @last
i=0.7*i(-1)+e

series data=NA

smpl @first @first+49
series data =t+s+i

smpl @first+50 @last
series data =t+s

smpl @all
graph aa data
show aa
```



Difference Equations and Their Solutions

- nth-order difference equation with constant coefficients

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t \tag{10}$$

where $x_t = \sum_{i=0}^{\infty} \beta_i \varepsilon_{t-i}$ and ε_t is a random disturbance term that has an expected value of zero

- A *solution* to a difference equation expresses the value of y_t as a function of the elements of the
  - $\{x_t\}$ sequence
  - t
  - initial conditions (y_0)

- So we have two considerations:

  1 How to solve linear difference equations?

  2 Whether that solution is stable or not?

Solution by Iteration

- Consider the first order homogeneous difference equation

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t \tag{17}$$

- Given the value of y_0, it follows that

$$\begin{aligned} y_t &= a_0 + a_1(a_0 + a_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= a_0 + a_1 a_0 + a_1^2 y_{t-2} + \varepsilon_t + a_1 \varepsilon_{t-1} \\ &= a_0 + a_1 a_0 + a_1^2(a_0 + a_1 y_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + a_1 \varepsilon_{t-1} \\ &= a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + \varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} \end{aligned}$$

$$\ldots$$

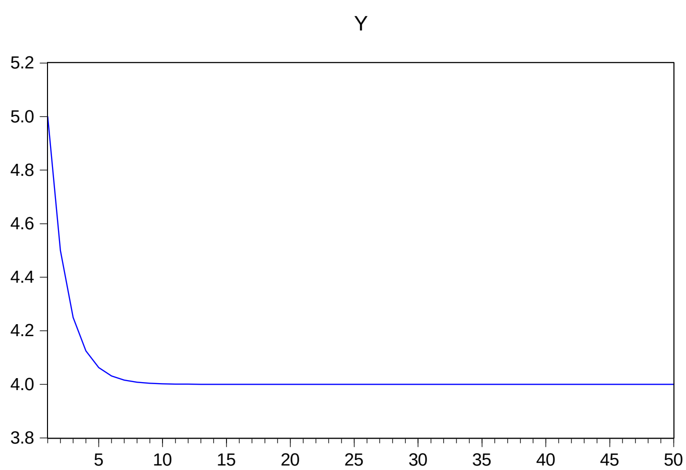
$$y_t = a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} \tag{18}$$

- If $|a_1| < 1$, the solution converges to

$$y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i} \tag{21}$$

E-views Application

```
wfcreate (wf=income process) u 50
series y=5

smpl @first+1 @last
y=0.5*y(-1)+2

smpl @all
graph aa y
show aa
```



- If $| \ a_1 \ |> 1$, then the $\{y_t\}$ series explodes, and the solution requires knowledge of initial conditions

$$y_t = a_0 \sum_{i=0}^{t-1} a_i^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_i^i \varepsilon_{t-i}$$

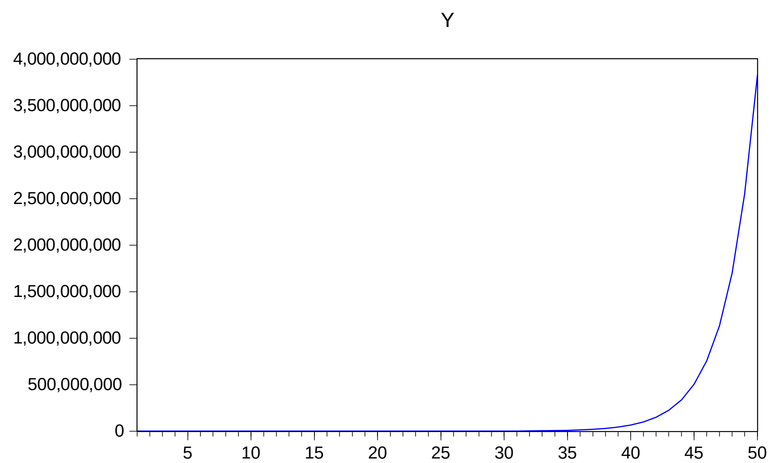
E-views Application

wfcreate (wf=income process) u 50

series y=5

smpl @first+1 @last
y=1.5*y(-1)+2

smpl @all
graph aa y
show aa



- If $a_1 = 1$, (17) is called a unit root process and its solution reduces to

$$y_t = a_0 t + \sum_{i=0}^{t} \varepsilon_i + y_0$$

  - $\sum_{i=1}^{t} \varepsilon_i$ is the random walk component
  - $a_0 t$ is the time trend
  - together, the $\{y_t\}$ series follow a random walk with a drift

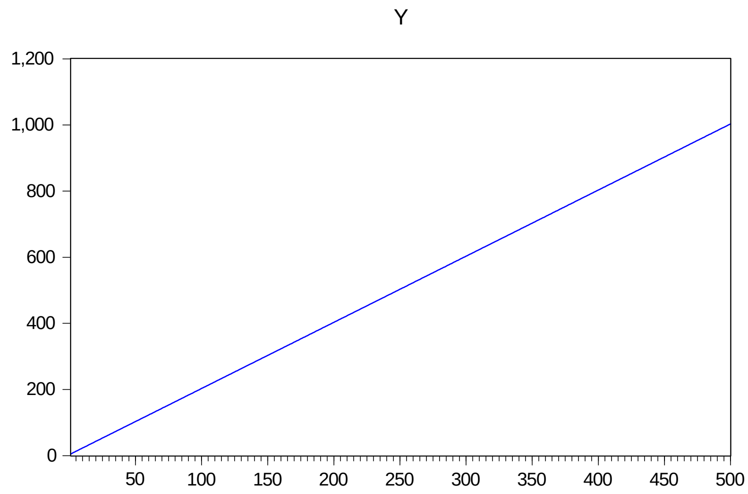
E-views Application (Drift Component-also called Trend Component)

wfcreate (wf=income process) u 500

series y=5

smpl @first+1 @last
y=y(-1)+2

smpl @all
graph aa y
show aa



E-views Application (Unit Root Component)

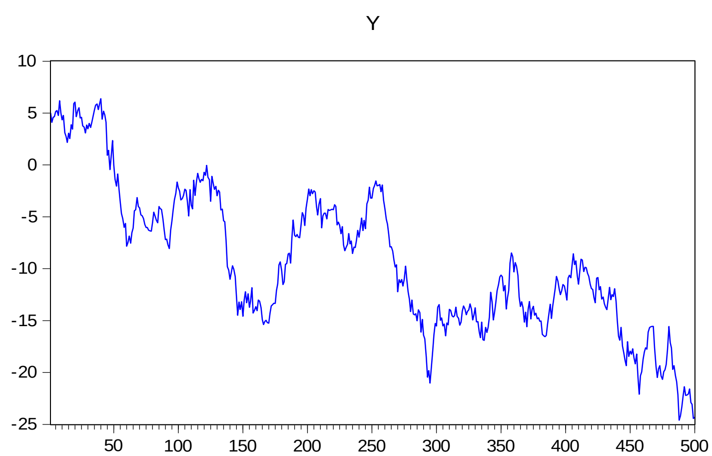
wfcreate (wf=income process) u 50

series y=5

series e=nrnd

smpl @first+1 @last
y=y(-1)+e

smpl @all
graph aa y
show aa



In this case, both the trend and the unit root components prevent the y_t series from converging to a stable point.

Why $\mid a_1 \mid = 1$ is Critical?

- Consider the equation

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t \tag{17}$$

- The homogeneous part of this equation is

$$y_t = a_1 y_{t-1}$$

its solution (which reflects the long-term dynamics of the model) is given by

$$y_t^h = a_1^t y_0$$

a_1 is called characteristic root of this equation

- If $\mid a_1 \mid > 1$, given y_0, the y_t series explodes as time goes to infinity

Solving Second Order Homogeneous Difference Equations

- Consider the homogeneous equation

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0 \tag{45}$$

- Its solution has the form

$$y_t^h = \alpha^t y_0$$

combining with equation (45)

$$\alpha^t y_0 - a_1 \alpha^{t-1} y_0 - a_2 \alpha^{t-2} y_0 = 0$$

dividing by α^{t-2}

$$\alpha^2 - a_1 \alpha - a_2 = 0 \tag{47}$$

- Solving this quadratic equation yields two characteristic roots:

$$\alpha_1, \alpha_2 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

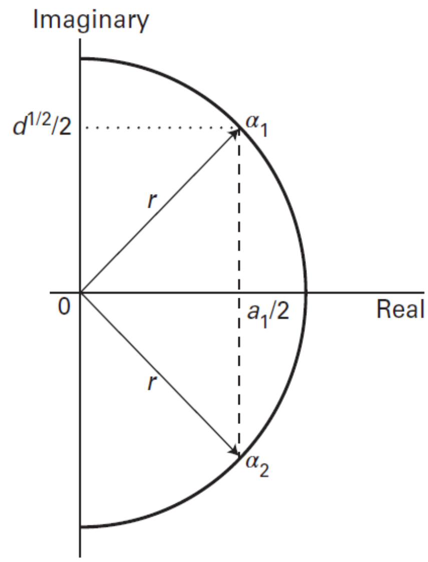
- If $a_1^2 + 4a_2 \geq 0$
  - There will be real characteristic roots
- If $a_1^2 + 4a_2 < 0$
  - In this case the characteristic roots have both real and imaginary parts

$$\alpha_1, \alpha_2 = (a_1 \pm i\sqrt{-d})/2$$

where $d = a_1^2 + 4a_2$ and $i = \sqrt{-1}$

Stability Conditions

- Consider the following semicircle



- Real numbers are measured on the horizontal axis and imaginary numbers are measured on the vertical axis
- Stability requires that all roots lie within a circle of radius one
- In this case the homogeneous solution will be convergent
  - If the characteristic roots are complex, the stability condition again requires that

$$r = \sqrt{(a_1/2)^2 + (i\sqrt{-d}/2)^2} < 1$$

- In the time-series literature, it is simply stated that stability requires that all characteristic roots lie within the unit circle.
  - If all characteristics roots lie within the unit circle, then the equation and its solution are stable
  - If at least one characteristics root lie outside the unit circle, then the equation and its solution are unstable
  - If at least one characteristics root lie on the unit circle, then the equation is unstable and contains a unit root

Example:

$$y_t = 0.2y_{t-1} + 0.35y_{t-2}$$

then

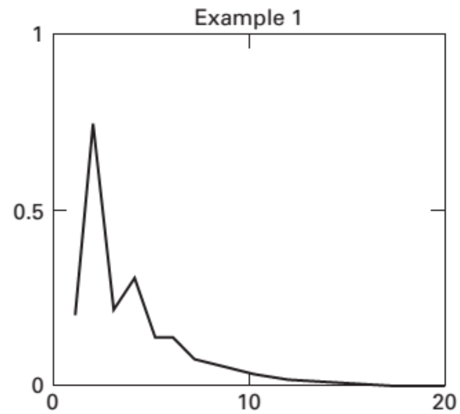
$$a_1 = 0.2 \quad \text{and} \quad a_2 = 0.35$$

$$\alpha_1, \alpha_2 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} = \frac{0.2 \pm \sqrt{0.2^2 + 4 * 0.35}}{2}$$

$$\alpha_1 = 0.7 \;\; \text{and} \;\; \alpha_2 = -0.5$$

- The homogeneous solution is

$$y_t = A_1(0.7)^t + A_2(-0.5)^t$$



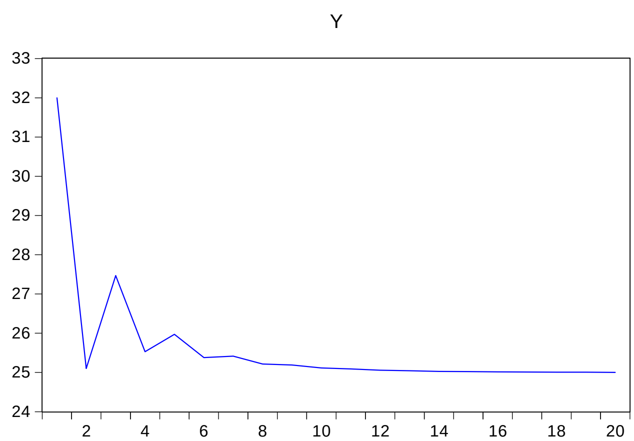
- Convergence is not monotonic because of the influence of the expression $(-0.5)^t$

E-views Application

wfcreate (wf=income process) u 20

series y=25+ 3*(0.7)^@TREND+ 4*(-0.5)^@TREND

graph aa y
show aa



Higher Order Systems

- See *Applied Econometric Time Series, Walter Enders* to check necessary and sufficient conditions for stability of higher order systems

Solution by Lag Operators

- The lag operator L is defined to be a linear operator such that for any value y_t

$$L^i y_t = y_{t-i}$$

- It has the following properties
  - The lag of a constant is constant
$$Lc = c$$
  - L raised to a negative power is actually a lead operator:
$$L^{-i} y_t = y_{t+i}$$

- Using lag operators, we can write the pth-order equation

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + ... + a_p y_{t-p} + \varepsilon_t$$

as

$$(1 - a_1 L - a_2 L^2 - ... - a_p L^p) y_t = a_0 + \varepsilon_t$$

or, more compactly as

$$A(L) y_t = a_0 + \varepsilon_t$$

where A(L) is the polynomial

- Lag operators can be used to express the equation

$$y_t = a_0 + a_1 y_{t-1} + ... + a_p y_{t-p} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + ... + \beta_q \varepsilon_{t-q}$$

as follows

$$A(L) y_t = a_0 + B(L) \varepsilon_t$$

- Consider the following first-order equation where $| a_1 |< 1$

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

- Using the definition of L

$$\begin{aligned}
(1 - a_1 L) y_t &= a_0 + \varepsilon_t \\
y_t &= \frac{a_0}{1 - a_1 L} + \frac{\varepsilon_t}{1 - a_1 L} \\
&= a_0 \sum_{i=0}^{\infty} (a_1 L)^i + \sum_{i=0}^{\infty} (a_1 L)^i \varepsilon_{t-i} \\
&= \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i} \qquad (21)
\end{aligned}$$

the last equation is the same with we found by iteration

Example

$$I_t = 0.7 I_{t-1} + \varepsilon_t$$

- Solution by iteration: Just substitute for the lags of I_t

$$I_t = 0.7(0.7I_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$
$$= 0.7^2 I_{t-2} + 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$= 0.7^2(0.7I_{t-3} + \varepsilon_{t-2}) + 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$...$$
$$= \sum_{i=0}^{\infty}(0.7)^i \varepsilon_{t-i}$$

- Solution by using the lag operator

$$I_t = 0.7LI_t + \varepsilon_t$$
$$I_t(1 - 0.7L) = \varepsilon_t$$
$$I_t = \frac{\varepsilon_t}{1 - 0.7L}$$
$$= \varepsilon_t + 0.7L\varepsilon_t + 0.7^2L^2\varepsilon_t + ...$$
$$= \sum_{i=0}^{\infty}(0.7)^i \varepsilon_{t-i}$$

Solving Second Order Homogeneous Difference Equations with Lag Operators

- Consider once again the homogeneous equation

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0 \tag{45}$$

which can be written as

$$y_t - a_1 L y_t - a_2 L^2 y_t = 0$$

which can be simplified as

$$1 - a_1 L - a_2 L^2 = 0$$

we can either try to solve this quadratic equation, or multiply it by L^{-2}, which finds

$$L^{-2} - a_1 L^{-1} - a_2 = 0$$

when we compare it with Equation (47)

$$\alpha^2 - a_1\alpha - a_2 = 0 \tag{47}$$

we see that $\alpha = L^{-1}$.

Example:

$$y_t = 0.2y_{t-1} + 0.35y_{t-2}$$

we know that the solution of this homogenous equation is

$$\alpha_1 = 0.7 \text{ and } \alpha_2 = -0.5$$

then the solution of the equation obtained with lag operators is

$$\alpha_1 = 1/0.7 \text{ and } \alpha_2 = -1/0.5 = -2$$

- Hence, when written in lag operators, stability requires that
  - If all characteristics roots lie outside the unit circle, then the equation and its solution are stable
  - If at least one characteristics root lie inside the unit circle, then the equation and its solution are unstable
  - If at least one characteristics root lie on the unit circle, then the equation is unstable and contains a unit root