Practice Problems 1

Problem 1: The Solow-Swan Model: Shares and Golden Rule

a-) Consider a Solow model without a technological growth. Define the Golden Rule of Capital Accumulation. What conditions determine the golden rule capital stock?

Solution: Golden Rule of Capital Accumulation is the amount of capital that maximizes consumption at the steady state.

We know that the steady state level of consumption is: $c^* = (1-s)f(k^*)$. The saving rate that maximizes this function can be found by

$$\max_s c^* = \max_s (1-s)f(k^*(s))$$

We have also shown that at the steady state: $sf(k^*) = (\delta + n)k^*$ (the investment is equal to the effective depreciation). Hence, the maximization problem can be written in terms of steady stae capital amount

$$\max_{k^*} c^* = \max_{k^*} [f(k^*) - (\delta + n)k^*]$$

Taking the FOC gives the condition for the Golden Rule of Capital Accumulation

$$f'(k_{gold}) = (\delta + n)$$

b-) Consider a Cobb Douglas technology with parameter $0 < \alpha < 1$. Imagine that the depreciation rate of capital is $\delta = 0.10$ and the rate of population growth is $n = 0.01$. Compute the golden rule capital stock k_{gold} as a function of α. What is the golden rule capital stock when $\alpha = 1/2$?

Solution: Using the condition found in part (a)

$$f'(k_{gold}) = (\delta + n)$$

and inserting for parameter values

$$f'(k^*_{gr}) = n + \delta = 0.01 + 0.1 = 0.11$$

using the production function

$$f(k^*_{gr}) = {k^*_{gr}}^{\alpha} \quad \Rightarrow \quad f'(k^*_{gr}) = \alpha {k^*_{gr}}^{\alpha - 1}$$

combining the last two equations: ${k^*_{gr}}^{\alpha-1} = \dfrac{0.11}{\alpha}$

$$\Rightarrow \quad k^*_{gr} = (\frac{0.11}{\alpha})^{1/(\alpha-1)} = (\frac{0.11}{1/2})^{1/(1/2-1)} = 0.22^{-2} = (1/0.22)^2 = (50/11)^2$$

c-) Imagine that the saving rate in the economy is the constant "s" (with $0 < s < 1$). What would be the value of s that delivers k_{gold} as the steady state capital stock assuming $\alpha = 1/2$?

Solution: At each s.s. of capital accumulation (either it is golden ruke or not) the change in capital stock is 0

$$\dot{k}_{gr}^{*} = sf(k_{gr}^{*}) - (n+\delta)k_{gr}^{*} = 0$$

which gives us the Golden Rule Level of Saving Rate

$$s = \frac{(n+\delta)k_{gr}^{*}}{{k_{gr}^{*}}^{\alpha}} = \frac{(n+\delta)}{{k_{gr}^{*}}^{\alpha-1}}$$

from part (b) we know that $\alpha {k_{gr}^{*}}^{\alpha-1} = (\delta + n)$. Thus

$$s = \frac{n+\delta}{(n+\delta)/\alpha} = \alpha = \frac{1}{2}$$

d-) Suppose now that inputs earn their marginal products. Show that when owners of capital save all their income and workers consume all their income, then the economy reaches the golden rule capital stock

Solution: We know from Cobb Douglas technology with competitive markets that inputs earn their marginal product: $Y = F_K K + F_L L$. This equation can also be written in per capita terms: $y = F_K k + F_L$. We have also shown that

$$\frac{\partial Y}{\partial K} = f'(k) \quad \text{and} \quad \frac{\partial Y}{\partial L} = f(k) - f'(k)k$$

When owners of capital save all their income and workers consume all their income, the total saving in the economy is

$$k * MPK = \frac{\partial (k^{\alpha})}{\partial k} = k\alpha k^{\alpha-1} = \alpha k^{\alpha} = \alpha y$$

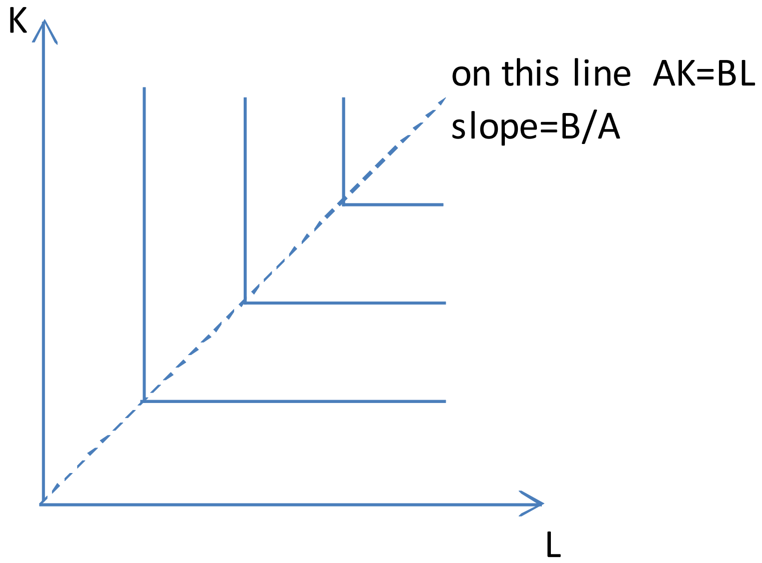
So α fraction of total output is saved. And this is the same amount we found in part (c) that is needed for each golden rule capital stock

Problem 2: Harrod-Domar

Consider the Solow-Swan model with constant savings rate, s. Imagine that the technology is Leontief so: $Y = \min(AK, BL)$ (1)

a-) Draw indifference curves in the $K - L$ space

Solution



b-) Write down $y = Y/L$ as a function of k

Solution

$$Y = \min(AK, BL) = L\min(Ak, B) \quad \Rightarrow \quad y = \begin{matrix} Ak & \text{if } k < B/A \\ B & \text{if } k > B/A \end{matrix}$$

c-) If the rate of population growth is zero ($n = 0$), show that the growth rate of capital is as follows:

$$\frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$$

Solution

$$\dot{K} = sY - \delta K \quad \Rightarrow \quad \dot{k} = sy - (n+\delta)k \quad \Rightarrow \quad \frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$$

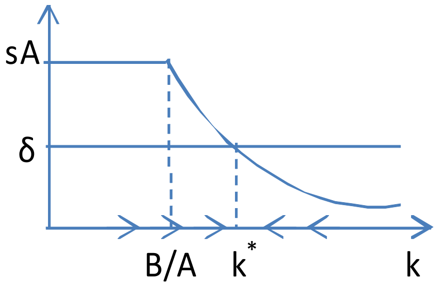
Draw the savings and the depreciation lines as a function of k. Find the steady state capital stock, k^*, for each of the cases written below. What are the dynamics of k over time? Do you see any problems with those steady states?

d-) $sA > \delta$ **e-)** $sA < \delta$ **f-)** $sA = \delta$

Solution: We know the dynamic equation for capital: $\frac{\dot{k}}{k} = \begin{cases} sA - \delta & k < B/A \\ sB/k - \delta & k > B/A \end{cases}$

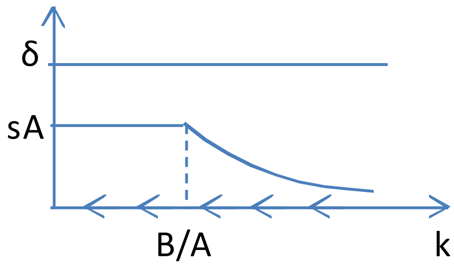
Hence,

$sA > \delta$



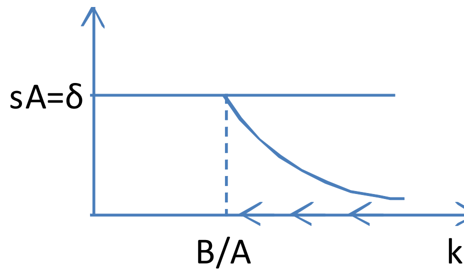
idle machines

$sA < \delta$



idle labor

$sA = \delta$



at $k^* = B/A$
without any idle factor