Practice Problems 2 (Ozan Eksi)

Problem 1 (Quadratic Utility Function and Fixed Income)

Let's assume that $\rho = r$, and consumers preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions together with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and assuming that $y_t = \bar{y}$,

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, \bar{y})$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty} (\frac{1}{1+r})^i E_t y_{t+i} + (1+r) A_t$$

and interpret your finding

Solution

Using $E_t c_{t+1} = c_t$ and $y_t = \bar{y}$ together with the PVBC defined above finds

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i \bar{y} + (1+r)A_t$$

since $r > 0$, it must be true that $\frac{1}{1+r} < 1$. Then the above equality can be written as

$$\frac{1}{1-\frac{1}{1+r}} c_t = \frac{1}{1-\frac{1}{1+r}} \bar{y} + (1+r)A_t$$

which can be simplified to

$$c_t = \bar{y} + rA_t$$

Interpretation: Since households earn fixed income each period and they have quadratic preferences for consumption, they do not save, and consume the same amount in each period, which is yearly labor income plus the yearly rate of return (annuity value) on wealth

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$, and interpret your findings

Solution

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (\bar{y} + rA_t) = 0$$

The transitory income is the saving, i.e. the difference between disposable income and the permanent income. If there is no saving, it must be equal to zero

Finding the relation between A_t and A_{t+1}

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (\bar{y} + rA_t) = A_t$$

Because households consume annuity value of their financial wealth, rA_t, the level of financial wealth in real terms does not change over periods, thus $A_{t+1} = A_t$

Problem 2: Quadratic Utility and Expected Permanent Change in Income
Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we use these two assumptions with the Euler Equation:

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and also assuming that

$$E_t(y_{t+i}) = \begin{cases} \bar{y} & i = 0 \\ 2\bar{y} & i = 1, 2, ...\infty \end{cases}$$

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, \bar{y})$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding

Solution: Using $E_t c_{t+1} = c_t$ and the defined time path of y_{t+i} together with the PVBC

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \bar{y} + \sum_{i=1}^{\infty}(\frac{1}{1+r})^i 2\bar{y} + (1+r)A_t$$

which can be written as

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i 2\bar{y} - \bar{y} + (1+r)A_t$$

since $r > 0$, it must be true that $\frac{1}{1+r} < 1$. As a result, the above equality can be written as

$$\frac{1}{1-\dfrac{1}{1+r}}c_t = \frac{1}{1-\dfrac{1}{1+r}}2\bar{y} - \bar{y} + (1+r)A_t$$

which can be simplified to

$$c_t = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t$$

Interpretation: Since households have quadratic preferences for consumption, they prefer to consume the same amount each period as long as there is no unexpected change in future income. Hence, even though the change in income is expected to take place in the future, the level of consumption is fixed over periods, and it is slightly less than $2\bar{y}$ (It is less because s/he gains $2\bar{y}$ on every period other than today. It is slightly less because assuming the real interest rate is small, $r/(1+r)$ is a small number)

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} under this case by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$ and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t) = -\bar{y} + \frac{r}{1+r}\bar{y} = -\frac{1}{1+r}\bar{y}$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (2\bar{y} - \frac{r}{1+r}\bar{y} + rA_t) = A_t - \frac{1}{1+r}\bar{y}$$

Interpretation: Since at time t income gain is less than the future incomes, households, which have quadratic preferences and like to smooth their consumption pattern, either borrow today or use some of their financial wealth to smooth their consumption. As a result we have found dissaving and that consumers financial wealth has declined over two time periods.

Problem 3: Quadratic Utility Function and Unexpected Permanent Change in Income

Let's assume that everything is the same with previous question, except that the consumer does not know at time t whether her/his income will change to $2\bar{y}$. So

$$E_t(y_{t+i}) = \bar{y} \qquad for \quad i = 1, 2, ...\infty$$

but the reality is

$$y_{t+i} = \begin{cases} \bar{y} & i = 0 \\ 2\bar{y} & i = 1, 2, ...\infty \end{cases}$$

and finally, once his income is increased to $2\bar{y}$ at time $t+1$, he changes his expectations as

$$E_{t+1}(y_{t+i}) = 2\bar{y} \qquad i = 2, ...\infty$$

a-) Find the difference between consumptions at time $t+1$ and at time t in terms of wealth and income of the consumers, $c_{t+1}(A_{t+1},\bar{y}) - c_t(A_t,\bar{y})$, by using the following Present Value Budget Constraint (Hint: you need to solve this constraint both at time t and $t+1$)

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding

Solution: Just like the 1st problem of problem session 2, if consumer thinks that s/he is going to earn $\bar{y}$, s/he consumes

$$c_t = \bar{y} + rA_t$$

when s/he updates her/his expectations, he starts to consume

$$c_{t+i} = 2\bar{y} + rA_t \qquad i = 2, ...\infty$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint at time t, $A_{t+1} = (1+r)A_t + y_t - c_t)$, and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (\bar{y} + rA_t) - (\bar{y} + rA_t) = 0$$

To find the relation between A_t and A_{t+1} we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + \bar{y} - (\bar{y} + rA_t) = A_t$$

Since saving equals to zero, the wealths at time t and $t+1$ are the same

Problem 4: Quadratic Utility Function and Expected Temporary Change in Income

Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Using this equality, and also assuming that

$$E_t(y_{t+i}) = \begin{cases} 2\bar{y} & i = 0, 1, 3, 4 ... \infty \\ \bar{y} & i = 2 \end{cases}$$

a-) Find consumption at time t in terms of wealth and income of the consumers, $c_t(A_t, y_t)$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your result

Solution: Use $E_t c_{t+1} = c_t$ and the condition for y_{t+i} defined above together with the PVBC

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_t = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i 2\bar{y} - \frac{\bar{y}}{(1+r)^2} + (1+r)A_t$$

since $r > 0$ and so $\frac{1}{1+r} < 1$. As a result, the above equality can be written as

$$\frac{1}{1-\frac{1}{1+r}}c_t = \frac{1}{1-\frac{1}{1+r}}2\bar{y} - \frac{\bar{y}}{(1+r)^2} + (1+r)A_t$$

which can be simplified as

$$c_t = 2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t)$), and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (2\bar{y} + rA_t) - (2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t) = \frac{r\bar{y}}{(1+r)^3}$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + 2\bar{y} - (2\bar{y} - \frac{r\bar{y}}{(1+r)^3} + rA_t) = A_t + \frac{r\bar{y}}{(1+r)^3}$$

Problem 5: Quadratic Utility Function and Unexpected Temporary Change in Income

Let's assume that $\rho = r$, and consumer preferences are represented by a quadratic utility function $u(c) = c - b/2 \cdot c^2$. When we used these two assumptions with the following Euler Equation

$$E_t u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t)$$

we found that $E_t c_{t+1} = c_t$. Now we use an individual who assumes that

$$E_t(y_{t+i}) = 2\bar{y} \qquad for \quad i = 0, 1, 2, ...\infty$$

but the reality is

$$y_{t+i} = \begin{cases} 2\bar{y} & i = 0, 2, 3...\infty \\ \bar{y} & i = 1 \end{cases}$$

and finally even though his income decreases to $\bar{y}$ at time $t+1$, he will realize that this is a temporary change and as a result he does not change his expectations about his future income

$$E_{t+1}(y_{t+2+i}) = 2\bar{y} \qquad i = 0, ...\infty$$

a-) Find the difference between consumptions at time $t+1$ and at time t in terms of wealth and income of the consumers, $c_{t+1}(A_{t+1}, y_{t+1}) - c_t(A_t, y_t)$, by using the following Present Value Budget Constraint

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t c_{t+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i E_t y_{t+i} + (1+r)A_t$$

and interpret your finding (Hint: you need to solve this constraint both at time t and $t+1$)

Solution: If consumer thinks that s/he is going to earn $\bar{y}$, s/he consumes

$$c_t = 2\bar{y} + rA_t$$

when s/he updates her/his expectations, he starts to consume

$$\sum_{i=0}^{\infty}(\frac{1}{1+r})^i c_{t+1+i} = \sum_{i=0}^{\infty}(\frac{1}{1+r})^i 2\bar{y} - \bar{y} + (1+r)A^*_{t+1}$$

$$c_{t+1} = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA^*_{t+1}$$

$$c_{t+i} = 2\bar{y} - \frac{r}{1+r}\bar{y} + rA_{t+i} \qquad i = 2, ...\infty$$

$$c_{t+1}(A_{t+1}, y_{t+1}) - c_t(A_t, y_t) = -\frac{r}{1+r}\bar{y}$$

b-) Find the transitory income y_t^T, and the relation between A_t and A_{t+1} by using the Budget Constraint $A_{t+1} = (1+r)A_t + y_t - c_t$, and interpret your findings

Solution: The transitory income is the saving, i.e. the difference between disposable income and the permanent income

$$y_t^T = y_t^D - y_t^P = y_t^D - c_t = (2\bar{y} + rA_t) - (2\bar{y} + rA_t) = 0$$

To find the relation between A_t and A_{t+1}, we can use the budget constraint

$$A_{t+1} = (1+r)A_t + y_t - c_t = (1+r)A_t + 2\bar{y} - (2\bar{y} + rA_t) = A_t$$

Problem 6 (Dynamic Programming)

Remember the Bellman Equation

$$V_t[(1+r)A_t + y_t] = \max_{c_t}\{u(c_t) + \frac{1}{1+\rho}E_t V_{t+1}[(1+r)A_{t+1} + y_{t+1}]\}$$

which is subject to

$$A_{t+1} = (1+r)A_t + y_t - c_t$$

Now suppose that there is no labor income and all income is derived from tradable wealth. Further suppose that $r = \rho$ and the utility of the consumer is of the functional form: $u(c) = c_t - b/2 \cdot c_t^2$. Use the Bellman equation to solve consumption, c, in terms of A

- Some additional Information:
  - All CRRA, CARA, and quadratic utility functions are the class of HRRA (Hyperbolic absolute risk aversion) utility function

– Merton shows that the value function is of the same functional form as the utility function for the HRRA utility functions if labor income is fully diversifiable, or where there is no labor income at all and all income is derived from tradable wealth

As a result, we use a following functional form for $V(\cdot)$

$$V_t(A_t) = aA_t^2 + dA_t + e$$

Then the Bellman equation

$$V_t(A_t) = \max_{c_t}\{u(c_t) + \frac{1}{1+\rho}E_t V_{t+1}(A_{t+1})$$

can be modified as (there is no uncertainty, so no need for E_t)

$$aA_t^2 + dA_t + e = \max_{c_t}\{c_t - b/2 \cdot c_t^2 + \frac{1}{1+\rho}[aA_{t+1}^2 + dA_{t+1} + e]$$

that is subject to

$$A_{t+1} = (1+r)A_t - c_t$$

FOC w.r.t. A_t gives

$$2aA_t + d = \frac{1+r}{1+\rho}(2aA_{t+1} + d)$$

Assuming $r = \rho$ and using the Budget Constraint

$$2aA_t + d = (2a[(1+r)A_t - c_t] + d)$$

which can be simplified to

$$A_t = [(1+r)A_t - c_t] = A_t - rA_t - c_t$$

then

$$c_t = rA_t$$

With quadratic utility and no labor income, households consume annuity value of their financial wealth

Note: Taking derivative only with respect to A_t but not c_t has turned out to be sufficient under this case

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